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The Cauchy process and the Steklov problem

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Abstract

Let X_t be a Cauchy process in $\mathbb{R}^d$, $d \geqslant 1$. We investigate some of the fine spectral theoretic properties of the semigroup of this process killed upon leaving a domain D. We establish a connection between the semigroup of this process and a mixed boundary value problem for the Laplacian in one dimension higher, known as the "*Mixed Steklov Problem.*" Using this we derive a variational characterization for the eigenvalues of the Cauchy process in D. This characterization leads to many detailed properties of the eigenvalues and eigenfunctions for the Cauchy process inspired by those for Brownian motion. Our results are new even in the simplest geometric setting of the interval $(-1, 1)$ where we obtain more precise information on the size of the second and third eigenvalues and on the geometry of their corresponding eigenfunctions. Such results, although trivial for the Laplacian, take considerable work to prove for the Cauchy processes and remain open for general symmetric α-stable processes. Along the way we present other general properties of the eigenfunctions, such as real analyticity, which even though well known in the case of the Laplacian, are not available for more general symmetric α-stable processes.
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1. Introduction

The potential theory for the symmetric α-stable processes, $0<\alpha<2$, in domains of Euclidean space has been extensively studied by many researchers for many years. In particular, many of the "fine" and now well-known results for Brownian motion ($\alpha = 2$) have been extended to these processes in recent years. These include, to name but a few, the boundary Harnack principles [9,12,45], the identification of the Martin boundary for various types of domains [10,23], the Harnack inequalities and conditional gauge theorems for α-stable Schrödinger semigroups [11,13,21,24], the notion of *intrinsic ultracontractivity* [21,36], sharp estimates for Green functions and Poisson kernels [22,42], and isoperimetric-type inequalities for heat kernels, Green functions, the lowest eigenvalue, and electrostatic capacities [2,5,39]. We refer the reader to [20] for a survey of some of these results. Despite the extensive literature on extension of these "fine" potential theoretic properties from the Brownian motion to the symmetric α-stable processes, many of the more detailed and refined spectral theoretic properties for which there is also an extensive literature in the case of Brownian motion (the Laplacian), remain completely open for general symmetric stable processes. This is the case even in the simplest geometric setting when the domain is the interval $(-1, 1)$. The purpose of this paper is to study some of these detailed properties for the eigenvalues and eigenfunctions in the case of the Cauchy process, $\alpha = 1$. Before we describe our result in more detail, and the reason why we need to restrict to the Cauchy process, we recall the basic definitions and some of the results for the Brownian motion which motivated the work presented in this paper.

Let X_t be a d-dimensional symmetric α-stable process of order $\alpha \in (0, 2]$ in $\mathbb{R}^d$. The process X_t has stationary independent increments and its transition density $p^\alpha(t,x,y) = p^\alpha(t,x-y),\ t>0,\ x,y \in \mathbb{R}^d$ is determined by its Fourier transform

$$\exp(-t|\xi|^\alpha) = \int_{\mathbb{R}^d} e^{i\xi\cdot y} p^\alpha(t,y)\,dy, \quad t>0,\ \xi \in \mathbb{R}^d.$$

That is, for any Borel subset $B \subset \mathbb{R}^d,\ x \in \mathbb{R}^d,\ t>0$,

$$P^x(X_t \in B) = \int_B p^\alpha(t,x,y)\,dy.$$

These processes have right continuous sample paths and their transition densities satisfy the following scaling property:

$$p^\alpha(t,x,y) = t^{-d/\alpha} p^\alpha(1, t^{-1/\alpha}x, t^{-1/\alpha}y).$$

When $\alpha = 2,\ X_t$ is just the usual d-dimensional Brownian motion B_t but running at twice the speed. That is, if $\alpha = 2$, then $X_t = B_{2t}$ and

$$p^2(t,x,y) = \frac{1}{(4\pi t)^{d/2}} \exp\left[\frac{-|x-y|^2}{4t}\right], \quad t>0,\ x,y \in \mathbb{R}^d.$$

When $\alpha = 1$, X_t is the Cauchy process in $\mathbb{R}^d$ whose transition densities are given by the Cauchy distribution (Poisson kernel)

$$p^1(t,x,y) = \frac{c_d t}{(t^2 + |x-y|^2)^{\frac{d+1}{2}}}, \quad t>0, \ x,y \in \mathbb{R}^d, \tag{1.1}$$

where

$$c_d = \Gamma\left(\frac{d+1}{2}\right) \Big/ \pi^{\frac{d+1}{2}}.$$

From this point on, unless otherwise clearly indicated, we assume that $\alpha = 1$. We will write $p(t,x,y)$ for $p^1(t,x,y)$. If $D \subset \mathbb{R}^d$ is a non-empty bounded open set, we let $\tau_D = \inf\{t \geqslant 0 : X_t \notin D\}$ be the first exit time of X_t from D and denote by P^x and E^x the associated probability and expectation for this process starting at x. We shall denote the semigroup on $L^2(D)$ of the Cauchy process killed upon leaving D by $\{P_t^D\}_{t \geqslant 0}$. That is, for $f \in L^2(D)$, $x \in D$, $t>0$,

$$P_t^D f(x) = E^x(f(X_t), \tau_D > t).$$

The semigroup has transition densities $p_D(t,x,y)$ and

$$P_t^D f(x) = \int_D p_D(t,x,y) f(y)\, dy.$$

The function $p_D(t,x,y)$ is positive symmetric and

$$p_D(t,x,y) \leqslant p(t,x,y) = \frac{c_d t}{(t^2 + |x-y|^2)^{\frac{d+1}{2}}} \leqslant \frac{c_d}{t^d}$$

for all $t>0$ and $x,y \in D$. In addition, for each fixed $t>0$, $p_D(t,x,y)$ is continuous on $D \times D$ as a functions of (x,y). We refer the reader to [21,36] for these elementary properties. It follows from this bound on the function $p_D(t,x,y)$ that for any open set D of finite volume, and in particular for any bounded set, the operator P_t^D generates a self-adjoint semigroup on $L^2(D)$ which is ultracontractive. That is, the operator P_t^D maps $L^2(D)$ into $L^\infty(D)$ for all $t>0$. Under these assumptions it follows from the general theory of heat semigroups [26] that there is an orthonormal basis of eigenfunctions $\{\varphi_n\}$ for $L^2(D)$ and corresponding eigenvalues $\{\lambda_n\}$ satisfying

$$0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots$$

with $\lambda_n \to \infty$ as $n \to \infty$. That is, the pair $\{\varphi_n, \lambda_n\}$ satisfies

$$P_t^D \varphi_n(x) = e^{-\lambda_n t} \varphi_n(x). \tag{1.2}$$

In addition, λ_1 is simple and the corresponding eigenfunction φ_1, often called the ground state eigenfunction, is strictly positive on D. By the continuity of the kernel in both variables x and y, the eigenfunctions φ_n are continuous and bounded. These general facts hold for all symmetric stable processes of index $0<\alpha\leqslant 2$. For more general properties of these semigroups, see [6,21,32].

The above construction is analogous to the construction for Brownian motion. If we replace the Cauchy process, $\alpha=1$, by the process associated with $\alpha=2$ (Brownian motion running at twice the speed) and assume in addition that D is connected and that ∂D is regular, then P_t^D is just the heat semigroup associated with the Laplacian in D with Dirichlet boundary conditions. In this case $p_D(t,x,y)$ is the fundamental solution of the heat equation in D, also called the heat kernel for D. Let us denote by $\{\psi_n,\mu_n\}_{n=1}^{\infty}$ the eigenfunctions and eigenvalues in this case. This pair is then the classical eigenfunction/eigenvalue solution of the Dirichlet Laplacian in D. That is, the pair satisfies

$$\begin{cases} \Delta\psi_n(x) = -\mu_n\psi_n(x), & x\in D, \\ \psi_n(x)=0, & x\in\partial D. \end{cases} \tag{1.3}$$

The Dirichlet eigenvalue problem (1.3) has been extensively studied for many years both analytically and probabilistically. It is well-known that geometric information on D, such as convexity, symmetry, volume growth, smoothness of its boundary, etc., provides information not only on the ground state eigenfunction ψ_1 and the ground state eigenvalue μ_1, but also on the spectral gap $\mu_2-\mu_1$, and on the geometry of the nodal domains of ψ_2. We recall here some of the classical results for the Laplacian which served as motivation for the investigations in this paper.

Recall that for any $f: D\to\mathbb{R}$, its nodal set is $f^{-1}\{0\}$ and a nodal domain of f is any connected component of $D\backslash f^{-1}\{0\}$. The celebrated Courant–Hilbert nodal domain theorem guarantees that ψ_n has no more than n nodal domains. In particular, ψ_2 has exactly two nodal domains. In [41], Payne proved that if D is a symmetric bounded convex domain in the plane, then the nodal line $N=\overline{\{x\in D: \psi_2(x)=0\}}$ for ψ_2 must intersect ∂D in exactly two points. He conjectured that such a result should hold for any planar convex domain, regardless of symmetry. This was proved by Melas [38] for bounded convex domains in the plane with smooth boundary (see also [1]). This kind of detailed information on the nodal line is crucial in proving $\mu_2-\mu_1>3\pi^2/d_D^2$ [3,27] for bounded planar convex domain of diameter d_D which are symmetric with respect to both coordinate axes and convex in both axes. Indeed, for such domains Payne [41] proved that the nodal line is one of the two axes of symmetry. For general convex domains in $\mathbb{R}^d$, an important result of Brascamp and Lieb [18] asserts that the eigenfunction ψ_1 is log concave. This result has had many interesting applications in the literature and in particular it can be used to prove that for general convex domain, $\mu_2-\mu_1>\pi^2/d_D^2$ [37,44]. (The general conjecture made in 1983 by van den Berg [4] that for any planar convex domain $\mu_2-\mu_1>3\pi^2/d_D^2$, remains open.) For many other applications of the Brascamp–Lieb log-concavity result, including applications to option pricing, and various other extensions, we refer the reader to Borell [14–16].

All of the above properties for the eigenvalues and eigenfunctions are completely unknown for general symmetric stable processes (or for the Cauchy process) even for the interval $(-1,1)$. Of course, various general results on the eigenvalues and eigenfunctions of the Cauchy process, and even for general symmetric stable processes, are known. For example: (1) A version of the celebrated Wyel's asymptotic law was proved in [6]. This asserts that if D is an open bounded non-empty set and $N(\lambda)$ denotes the number of eigenvalues which are smaller than or equal to λ, and $m(\partial D)=0$, then $N(\lambda)\approx\lambda^d m(D)c_d(\Gamma(1+d))^{-1}$ as $\lambda\to\infty$ [6]. (2) It was proved recently that for all bounded domains the semigroup P_t^D is intrinsically ultracontractive [21,36]. Intrinsic ultracontractivity is closely related to the parabolic boundary Harnack principle and to conditioned processes (the associated Doob h-processes). It gives very sharp estimates on φ_n in terms of φ_1 and we will indeed use some of these estimates below, (see (2.6) in Section 2). In addition, if ∂D is suitably smooth then $\varphi_1(x)$ behaves like $(\operatorname{dist}(x,\partial D))^{1/2}$. (3) It is known that among all domains of fixed volume the ball has the smallest λ_1 (the Faber–Krahn inequality) and that among all convex domains of inradius R_D (the radius of the largest ball contained in D) λ_1 is minimized by the infinite strip and maximized by the ball of radius R_D. We refer the reader to [2,39] for many other "isoperimetric-type" results for general symmetric stable processes.

While an explicit expression for λ_1 is not known even for the interval $(-1,1)$, the comparison estimates in [2,39] lead to explicit upper and lower bounds for λ_1 for $(-1,1)$. As far as estimates on $\lambda_n,\ n\geqslant 2$, and geometric properties of $\varphi_n,\ n\geqslant 2$ for $(-1,1)$ are concerned, nothing seems to be known. Indeed, it was this very simple geometric situation which initially motivated our investigations that led to this paper. We were particularly interested in obtaining bounds for $\lambda_1,\ \lambda_2,\ \lambda_2-\lambda_1$ and geometric properties for φ_1 and φ_2 for $(-1,1)$. It may be proved (Section 4) that there exists an eigenfunction which is antisymmetric and (up to a sign) negative on $(-1,0)$ and positive on $(0,1)$. One of the first goals of this paper was to prove that this is the second eigenfunction. Unlike the case of the Laplacian, the proof is not easy. This is due in part to the fact that the Courant–Hilbert nodal line theorem is not known for operators which are not local. We succeeded in obtaining properties for φ_2 and λ_2 for $(-1,1)$ because of the connection of the Cauchy process to the Steklov problem. We will now describe this connection.

The central difficulty from the analytic point of view in studying some of the fine properties of λ_n and φ_n for the semigroup P_t^D is that its infinitesimal generator, A_D, is not a local differential operator. We may define A_D formally by

$$A_D f=\lim_{t\downarrow 0}\frac{P_t^D f-f}{t} \tag{1.4}$$

for such $f\in L^2(D)$ for which this limit exists in $L^2(D)$. The set of such functions (the domain of A_D) is denote by $\mathscr{D}(A_D)$. Similarly we define $A_D f(x)=\lim_{t\downarrow 0}(P_t^D f(x)-f(x))/t$ for any $f\in C(D)$ and $x\in D$ for which the limit exists. It may be shown that for $f\in C_c^2(D)$ and $x\in D$, $A_D f(x)$ is well defined and we have $A_D f(x)=-(-\Delta)^{1/2}f(x)$.

The definition of $(-\Delta)^{1/2}$ may be find for example in [11] (Definition 3.2, Lemma 3.5). We want to emphasize that we will not use the operator $(-\Delta)^{1/2}$ in any essential way in this paper, we just want to present the connection between the semigroup P_t^D and the operator $(-\Delta)^{1/2}$.

The expression

$$\mathscr{E}(f,g) = -\langle A_D f, g\rangle = -\int_D (A_D f) g\,dx \tag{1.5}$$

defines a Dirichlet form with domain $\mathscr{D}(\mathscr{E}) \subset L^2(D)$ [31, Theorem 1.3.1, Corollary 1.3.1], here $f \in \mathscr{D}(A_D)$, $g \in \mathscr{D}(\mathscr{E})$.

It is well-known that $\varphi_n \in \mathscr{D}(A_D)$, $(-\Delta)^{1/2}\varphi_n(x)$, $A_D\varphi_n(x)$ are well defined for $x \in D$ and $A_D\varphi_n(x) = -(-\Delta)^{1/2}\varphi_n(x) = -\lambda_n\varphi_n(x)$, $x \in D$. With this, we may write an analog of (1.3) with Δ replaced by $-(-\Delta)^{1/2}$. However, due to the non-locality of this operator it is difficult to use this representation to study the influence of the geometry of D on φ_n and on λ_n. The main idea in this paper, and the reason why we need to restrict our attention to the case $\alpha = 1$, is based on the connection between the eigenvalue problem for the Cauchy process and a mixed boundary eigenvalue problem for the Laplacian in one dimension higher, known as the "*mixed Steklov*" problem. Probabilistically, this amounts to thinking of the Cauchy processes as the trace of Brownian motion in one dimension higher. This idea will help us avoid dealing with $(-\Delta)^{1/2}$ and the difficulties related to the non-locality of this operator. However, even for bounded domains $D \subset \mathbb{R}^d$ the boundary value problem that arises takes place in unbounded domains and has not, as far as we know, been treated in the literature. Hence, we must deal with many basic questions and estimates for this problem.

The connection between our eigenvalue problem (1.2) and the Steklov problem arises as follows. For $f \in L^1(\mathbb{R}^d)$ we set

$$P_t f(x) = \int_{\mathbb{R}^d} p(t,x,y) f(y)\,dy,$$

where $p(t,x,y)$ is given by (1.1). For $f \in L^2(D)$ we extend it to all of $\mathbb{R}^d$ by putting $f(x) = 0$ for $x \in D^c$. Since D is bounded we see that such functions are also in $L^1(\mathbb{R}^d)$. Thus $P_t f(x)$ is well defined for $f \in L^2(D)$ by our bound on $p(t,x,y)$ and in particular it is well defined for any eigenfunction φ_n of our eigenvalue problem (1.2) extended to be zero outside of D. For any $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ and $t > 0$ we put

$$u_n(x,t) = P_t\varphi_n(x) \quad \text{and} \quad u_n(x,0) = \varphi_n(x). \tag{1.6}$$

This defines a function in

$$H = \{(x,t) : x \in \mathbb{R}^d,\ \ t \geqslant 0\}.$$

Since φ_n is continuous at least on $\mathbb{R}^d \backslash \partial D$ the function u_n is continuous at least on $H \backslash \{(x,0): x \in \partial D\}$. For many "regular domains" such as bounded Lipschitz domains, φ_n is continuous on all of $\mathbb{R}^d$ (see (3.2)), so that u_n is continuous on all of H. We will denote by H_+ the interior of the set H. That is, $H_+ = \{(x,t): x \in \mathbb{R}^d,\ t > 0\}$. Let

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial t^2}$$

denote the Laplace operator in H_+.

Theorem 1.1. *Let $D \subset \mathbb{R}^d$ be a bounded domain. Then*

$$\Delta u_n(x,t) = 0, \quad (x,t) \in H_+, \tag{1.7}$$

$$\frac{\partial u_n}{\partial t}(x,0) = -\lambda_n u_n(x,0), \quad x \in D \tag{1.8}$$

$$u_n(x,0) = 0, \quad x \in D^{\mathrm{c}}. \tag{1.9}$$

The idea of transforming problems for the non-local generator of symmetric α-stable processes to problems for local operator in $\mathbb{R}^{d+1}$ has been used in the past, see for example [28,40]. This idea was also used in a very general context in [43].

If Ω is a bounded domain in $\mathbb{R}^d$ and we write its boundary $\partial\Omega$ as the disjoint union of two pieces, $(\partial\Omega)_1$ and $(\partial\Omega)_2$ then the classical "*mixed Steklov*" *eigenvalue problem* [29,30,34] is the following mixed boundary value problem:

$$\Delta u_n(z) = 0, \quad z \in \Omega, \tag{1.10}$$

$$\frac{\partial u_n}{\partial v}(z) = -e_n u_n(z), \quad z \in (\partial\Omega)_1, \tag{1.11}$$

$$u_n(z) = 0, \quad z \in (\partial\Omega)_2, \tag{1.12}$$

where $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ and $\frac{\partial}{\partial v}$ is the inner normal derivative. The basic difference between our Steklov problems and the classical one in that our domain is unbounded.

The transformation of our eigenvalue problem (1.2) for the Cauchy process to (1.7)–(1.9) enables us to use variational methods, and in particular to derive a variational formula for λ_n (Theorem 3.8) and to prove an analog of the Courant–Hilbert nodal domain theorem (Theorem 3.11). Under some additional assumptions on D, we will also show that $\lambda_n \leqslant \sqrt{\mu_n}$. A comparison result of this type for all $0 < \alpha < 2$ was proved in [2] for λ_1. In addition, we obtain various other results for the eigenvalues and eigenfunctions of the Cauchy process from the corresponding Steklov problem.

The paper is organized as follows. In Section 2 we set some notation and present various known facts for the Cauchy process which are needed in the sequel. We also obtain a new upper bound estimate on λ_1 for balls in $\mathbb{R}^d$ which holds for all $0<\alpha<2$. In particular, if $D=(-1,1)$ and $\alpha=1$ we have $1\leqslant\lambda_1\leqslant 3\pi/8$. This estimate is better than the previous best bounds contained in [2]. In Section 3, we establish the connection between the Cauchy eigenvalue problem and the *mixed Steklov* boundary eigenvalue problem and prove the variational characterization for λ_n.

In Section 4, we prove several results based on properties of the transition density $p_D(t,x,y)$. One of the main result in this section (Theorem 4.3) asserts that whenever D is symmetric relative to one of the coordinate axis, then there exists an antisymmetric eigenfunction which is positive on the portion of D which lies on one side of the axis and negative on the portion of D which lies on the other side. It comes as a surprise to us that such results are essentially trivial for the Brownian motion (the Dirichlet Laplacian) but not so for the Cauchy processes. The basic idea for this argument is to use the multiple integral representation of the kernel coming from the semigroup property to construct a new semigroup.

In Section 5, we use the some of the results obtained in the previous sections to perform a much more detailed study for the Cauchy eigenvalues and eigenfunctions on what is perhaps the simplest geometric setting for these type of problems, the interval $D=(-1,1)$. We will show that φ_1 is symmetric and concave on $(-1,1)$ (see Theorem 5.1). It is in fact non-decreasing on $(-1,0)$ and non-increasing on $(0,1)$, and hence it satisfies the Brascamp–Lieb [18] concavity result. However, the main result of this section deals with geometric properties of φ_2 and λ_2. We shall prove that $2\leqslant\lambda_2\leqslant\pi$ and that its corresponding eigenfunction φ_2 is antisymmetric and (up to sign) negative on $(-1,0)$ and positive on $(0,1)$ (Theorem 5.3), similar to the situation for the Brownian motion. From this it will follow that φ_2 has two nodal domains and one nodal set. Moreover, we will show that φ_2 is concave on $(0,1)$ and convex on $(-1,0)$. In this section we also obtain various properties for λ_3 and φ_3 (Theorem 5.4). Furthermore, an application of our Courant–Hilbert nodal domain theorem for the Cauchy process proved in Section 3 will give that φ_n, $n\geqslant 1$, has at most $2n-2$ zeros in $(-1,1)$. This implies that φ_n has at most $2n-1$ nodal domains. Again, we find it remarkable that these properties, as simple as they are for Brownian motion, take considerable work to prove for the Cauchy process and that, outside of $\alpha=1$ and 2, they remain unknown for other symmetric α-stable processes.

2. Preliminary results

In this section we introduce some more notation, prove Theorem 1.1 and obtain some new bounds on the ground state eigenvalue. These bounds hold for all $0<\alpha<2$ and are of independent interest. Let $\mathbb{N}=\{1,2,\dots\}$ denote the set of natural numbers. For $d\in\mathbb{N}$, we denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^d$. For any subset

$U \subset \mathbb{R}^d$ we use U^c, $\bar{U}$, $\text{int}(U)$, and ∂U to denote its complement, closure, interior, and boundary, respectively. Furthermore, for $x \in \mathbb{R}^d$, $r > 0$ and $U, V \subset \mathbb{R}^d$, we put $B(x,r) = \{y \in \mathbb{R}^d : |x-y| < r\}$, $rU = \{ry : y \in U\}$, $\text{dist}(U,V) = \inf\{|y-z| : y \in U, z \in V\}$ and $\delta_U(x) = \text{dist}(x, \partial U)$. By $\mathscr{B}(\mathbb{R}^d)$ we mean the Borel σ-algebra of $\mathbb{R}^d$. We will write $c = c(\alpha, \beta, \ldots, \gamma)$ to indicate the dependence of a constant c on the parameters indicated. The constants may change their value from one use to the next and even on the same line in the same formula. However, the set of parameters on which a constant may depend will not change from one use to the another. The constants denoted with c will always be assumed to be finite and positive.

By a *domain* $D \subset \mathbb{R}^d$ we shall mean an open non-empty set. For $d \geqslant 2$ a bounded domain $D \subset \mathbb{R}^d$ is called a *bounded Lipschitz domain* if there exists a Lipschitz constant $M = M(D) > 0$ and a localization radius $r_0 = r_0(D) > 0$ satisfying the following property: For every $Q \in \partial D$ there is a Lipschitz function $\Gamma_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ of constant no worst than M and an orthonormal coordinate system CS_Q such that if $y = (y_1, \ldots, y_{d-1}, y_d)$ in the CS_Q coordinates, then

$$D \cap B(Q, r_0) = \{y : y_d > \Gamma_Q(y_1, \ldots, y_{d-1})\} \cap B(Q, r_0).$$

For completeness, a bounded Lipschitz domain on the real line ($d = 1$) is the union of a finite number of disjoint bounded open intervals with no common endpoints. Notice that, unlike the usual definition, we do not assume that D is necessarily connected. In dimensions $d \geqslant 2$ a bounded domain $D \subset \mathbb{R}^d$ is called a *bounded C^∞ domain* if it satisfies the same conditions as the bounded Lipschitz domain where the Lipschitz function is replaced by C^∞-function. The definition of a bounded C^k domain for any $k \geqslant 1$, is analogous. A bounded C^∞ domain or bounded C^k domain on the real line is the same as the bounded Lipschitz domain.

As above, we denote the transition probabilities for the killed process in the bounded domain D by $p_D(t,x,y)$. A probabilistic representation for this kernel is given by

$$p_D(t,x,y) = p(t,x,y) - r_D(t,x,y),$$

for $t > 0$, $x, y \in D$, where

$$r_D(t,x,y) = E^x(\tau_D < t; p(t - \tau_D, X(\tau_D), y)). \tag{2.1}$$

We now recall some other useful properties related to the Cauchy semigroup. For $x, y \in D$ let

$$G_D(x,y) = \int_0^\infty p_D(t,x,y)\, dt$$

and recall that

$$P^x\{\tau_D > t\} = \int_D p_D(t, x, y) dy.$$

Hence,

$$E^x(\tau_D) = \int_0^\infty P^x(\tau_D > t)\, dt = \int_0^\infty \int_D p_D(t, x, y)\, dy\, dt = \int_D G_D(x, y)\, dy.$$

We call $G_D(x, y)$ the Green function for D. This gives rise to the Green operator $G_D : L^2(D) \to L^2(D)$ defined by

$$\begin{aligned} G_D f(x) &= \int_D G_D(x, y) f(y)\, dy \\ &= \int_0^\infty E^x\{f(X_t; \tau_D > t\}\, dt = E^x\left(\int_0^{\tau_D} f(X_t)\, dt\right) \end{aligned}$$

for all $x \in D$, $f \in L^2(D)$. We note in particular that

$$G_D \varphi_n(x) = \varphi_n(x)/\lambda_n \tag{2.2}$$

for all $n \in \mathbb{N}$. In addition,

$$||G_D||_{2\to 2} = 1/\lambda_1, \tag{2.3}$$

where $||G_D||_{2\to 2}$ denotes the operator norm on $L^2(D)$. It follows from [3] that for all domains D of finite volume, $E^x(\tau_D) \leqslant E^0\left(\tau_D^*\right)$, where D^* is the ball of same volume as D. In particular, for bounded domain, $E^x(\tau_D) \in L^p(D)$ for any $0 < p \leqslant \infty$. It is also well-known that the function $u(x) = E^x(\tau_D)$ is in the domain of A_D as defined in (1.4) and that $A_D u(x) = -1,\ x \in D$.

In addition to the above properties, the semigroup P_t^D shares many other important properties with the semigroup of Brownian motion and in some instances is better behave. In particular, for all bounded domains $D \subset \mathbb{R}^d$ our Cauchy semigroup is *intrinsically ultracontractive.* (This is not the case for Brownian motion.) That is, for all $\varepsilon > 0$ there exists a constant $c = c(\varepsilon, D)$ such that for all $t > c$ and all $x, y \in D$,

$$(1 - \varepsilon) e^{-\lambda_1 t} \varphi_1(x) \varphi_1(y) \leqslant p_D(t, x, y) \leqslant (1 + \varepsilon) e^{-\lambda_1 t} \varphi_1(x) \varphi_1(y). \tag{2.4}$$

We also have

$$c_1(D) E^x(\tau_D) \leqslant \varphi_1(x) \leqslant c_2(D) E^x(\tau_D) \tag{2.5}$$

for all $x \in D$ and

$$|\varphi_n(x)| \leqslant c(t, D) e^{\lambda_n t} \varphi_1(x), \tag{2.6}$$

for all $t>0$, $x\in D$, $n\in\mathbb{N}$. Inequality (2.6) follows directly from (2.4) as does the right-hand side of (2.5). We refer the reader to [36] for full details on these results. Another result which we will need in the sequel asserts that if $D\subset\mathbb{R}^d$ is a bounded Lipschitz domain, then there exist constants $\beta_1=\beta_1(D)\in(0,1)$, $\beta_2=\beta_2(D)\in(0,1)$, $c_1(D,\beta_1)$ and $c_2(D,\beta_2)$ such that

$$c_1(D,\beta_1)\delta_D^{\beta_1}(x)\leqslant E^x(\tau_D)\leqslant c_2(D,\beta_2)\delta_D^{\beta_2}(x),\quad x\in D \tag{2.7}$$

and hence the same is true for the eigenfunction φ_1 by (2.5). The proof of (2.7) uses the Ikeda–Watanabe formula [35] and boundary Harnack principle techniques, see [9, Lemmas 3, 5]; [19, (2.9)].

The eigenvalues λ_n also satisfy the following useful scaling property: For any $\gamma>0$ we have $\lambda_n(\gamma D)=\lambda_n(D)/\gamma$, where $\lambda_n(D)$ is the eigenvalue for D and $\lambda_n(\gamma D)$ is the eigenvalue for γD.

Proof of Theorem 1.1. Formula (1.7) follows from the fact that the Poisson kernel for the half-space H, (where $(x,t)\in H_+$, $(y,0)\in\partial H$), is just $p(t,x,y)$ given by (1.1). Eq. (1.9) is obvious by the definition of $u_n(x,0)$ in (1.6). It remains to show (1.8). For $x\in D$ we have

$$\frac{\partial u_n}{\partial t}(x,0)=\lim_{t\to0^+}\frac{P_t\varphi_n(x)-\varphi_n(x)}{t}$$

and

$$\begin{aligned}\frac{P_t\varphi_n(x)-\varphi_n(x)}{t}&=\frac{P_t^D\varphi_n(x)-\varphi_n(x)}{t}+\frac{P_t\varphi_n(x)-P_t^D\varphi_n(x)}{t}\\&=\frac{(e^{-\lambda_n t}-1)\varphi_n(x)}{t}+\frac{1}{t}\int_D r_D(t,x,y)\varphi_n(y)\,dy,\end{aligned}$$

where $r_D(t,x,y)$ is given by (2.1). Clearly $(e^{-\lambda_n t}-1)/t\to-\lambda_n$ when $t\to0^+$ so to prove (1.8) it is sufficient to show that for each $x\in D$,

$$\lim_{t\to0^+}\frac{1}{t}\int_D r_D(t,x,y)|\varphi_n(y)|\,dy=0. \tag{2.8}$$

By (2.1) and the fact that $r_D(t,x,y)=r_D(t,y,x)$ we obtain for any $t>0$, $x,y\in D$, that $r_D(t,x,y)$ is equal to $E^y(\tau_D<t;p(t-\tau_D,X(\tau_D),x))$. Hence

$$\begin{aligned}\frac{1}{t}r_D(t,x,y)&=\frac{1}{t}E^y\left(\frac{c_d(t-\tau_D)}{((t-\tau_D)^2+|X(\tau_D)-x|^2)^{\frac{d+1}{2}}};\tau_D<t\right)\\&\leqslant\frac{1}{t}E^y\left(\frac{c_d t}{\delta_D^{d+1}(x)};\tau_D<t\right)=\frac{c_d P^y(\tau_D<t)}{\delta_D^{d+1}(x)}.\end{aligned} \tag{2.9}$$

When $t \to 0^+$ the last expression tends to 0. Since $\varphi_n \in L^\infty(D)$, we get (2.8) by the bounded convergence theorem. □

In Section 3 below we will present, as an application of our variational formulas, upper bounds estimates for the Cauchy eigenvalues in terms of the eigenvalues for the Laplacian. From these and our knowledge of the eigenvalues of the Laplacian, one can obtain estimates on the eigenvalues of the Cauchy processes. This idea was used in [2] to give estimates on the first eigenvalue of α-symmetric stable processes for various domains. For the unit ball in $\mathbb{R}^d$, the following proposition can be used to improve upon the upper bound of [2].

Proposition 2.1. *Let $D \subset \mathbb{R}^d$ be a bounded domain. We have*

$$\frac{1}{\sup_{x \in D} E^x(\tau_D)} \leqslant \lambda_1 \leqslant \frac{\int_D E^x(\tau_D)\, dx}{\int_D [E^x(\tau_D)]^2\, dx}. \tag{2.10}$$

Proof. The lower bound is well known. In fact,

$$\frac{\varphi_1(x)}{\lambda_1} = E^x\left(\int_0^{\tau_D} \varphi_1(X_t)\, dt\right) \leqslant ||\varphi_1||_\infty E^x(\tau_D). \tag{2.11}$$

For the upper bound we use the following equality [31, Lemma 1.5.3, p. 33] valid for any Dirichlet forms. For any non-negative $f \in L^1(D) \cap L^2(D)$,

$$\sup_{u \in \mathscr{D}(\mathscr{E})} \frac{\int_D f\, |u|\, dx}{\sqrt{\mathscr{E}(u,u)}} = \sqrt{\int_D f G_D f\, dx}.$$

This, Schwarz inequality and (2.3) gives that for all $u \in \mathscr{D}(\mathscr{E})$,

$$\begin{aligned}
\int_D f|u|\, dx &\leqslant \sqrt{\mathscr{E}(u,u)}\sqrt{\int_D f G_D f\, dx} \\
&\leqslant \sqrt{\mathscr{E}(u,u)}\sqrt{||f||_2 ||G_D f||_2} \\
&\leqslant \sqrt{\mathscr{E}(u,u)}\sqrt{\frac{1}{\lambda_1}||f||_2 ||f||_2} \\
&= ||f||_2 \sqrt{\mathscr{E}(u,u)}\sqrt{\frac{1}{\lambda_1}}.
\end{aligned}$$

Taking $u(x) = f(x) = E^x(\tau_D)$ and observing, as we did earlier, that $\sqrt{\mathscr{E}(u,u)} = \sqrt{\langle 1, u\rangle}$ gives the right-hand side of the proposition. □

We note that the above estimates for λ_1 hold for all $0<\alpha\leqslant 2$. For $\alpha=2$ the upper estimate follows from the variational formula for λ_1 and integration by parts.

Let us look at the special case of the ball $B(0,1)$ in $\mathbb{R}^d$. (For a general ball of radius r similar estimates follow trivially by scaling.) For any $0<\alpha\leqslant 2$ we have by Getoor [33]

$$E^x(\tau_{B(0,1)}) = C_{\alpha,d}\left(1-|x|^2\right)^{\alpha/2}, \tag{2.12}$$

where

$$C_{\alpha,d} = \frac{\Gamma(\frac{d}{2})}{2^\alpha \Gamma(1+\frac{\alpha}{2})\Gamma(\frac{d+\alpha}{2})}.$$

Clearly

$$\sup_{x\in B(0,1)} E^x(\tau_{B(0,1)}) = C_{\alpha,d}$$

and a simple integration in polar coordinates gives

$$\int_{B(0,1)} E^x(\tau_{B(0,1)})\,dx = \frac{C_{\alpha,d}\sigma_d}{2} B\left(\frac{d}{2},\frac{\alpha}{2}+1\right)$$

and

$$\int_{B(0,1)} [E^x(\tau_{B(0,1)})]^2\,dx = \frac{C_{\alpha,d}^2\sigma_d}{2} B\left(\frac{d}{2},\alpha+1\right),$$

where σ_d is the surface area of the unit sphere in $\mathbb{R}^d$ ($\sigma_1=2$) and

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is the Beta function. These calculations and Proposition 2.1 give

Corollary 2.2. *Let $\lambda_{\alpha,d}$ be the smallest eigenvalue for the symmetric stable process of order $0<\alpha\leqslant 2$, killed off $B(0,1)\subset\mathbb{R}^d$. Then*

$$\frac{1}{C_{\alpha,d}} \leqslant \lambda_{\alpha,d} \leqslant \frac{1}{C_{\alpha,d}} \frac{B(\frac{d}{2},\frac{\alpha}{2}+1)}{B(\frac{d}{2},\alpha+1)}. \tag{2.13}$$

In particular, for the Cauchy process in unit interval $(-1,1)$ in $\mathbb{R}$ and the unit disk in $\mathbb{R}^2$ we have, respectively,

$$1 \leqslant \lambda_{1,1} \leqslant \frac{3\pi}{8} \approx 1.178 \tag{2.14}$$

and

$$1.57 \approx \frac{\pi}{2} \leqslant \lambda_{1,2} \leqslant \frac{2\pi}{3} \approx 2.094. \tag{2.15}$$

Both (2.14) and (2.15) follow from (2.13) by a simple calculation, we leave it to the reader. The left-hand side of (2.13) is already in [2]. However, the right-hand side gives better estimates, at least in dimensions one and two, than those that follow from [2].

3. The mixed Steklov problem

In this section we exploit the connection of the eigenvalue problem for the Cauchy process to the Steklov problem described by Theorem 1.1. Using this we obtain a variational characterization for the eigenvalues λ_n. Many of these results, such as the variational formulas, are known for the Steklov problem (1.10)–(1.12) in bounded smooth domains Ω. Obtaining this results for our problem (1.7)–(1.9) for the unbounded domain $H_+ \subset \mathbb{R}^{d+1}$ when $D \subset \mathbb{R}^d$ is a bounded Lipschitz domain requires close attention to several technical details.

Proposition 3.1. *Let $D \subset \mathbb{R}^d$ be a bounded domain. Let $u_n(x,t)$ be as in* (1.6). *For $x \in \operatorname{int}(D^c)$, set*

$$r_n(x) = \lim_{t \to 0^+} \frac{u_n(x,t)}{t}$$

and $r_n(x) = 0$ for $x \in \bar{D}$. Then $r_n(x)$ is well defined for all $x \in \mathbb{R}^d$ and for $x \in \operatorname{int}(D^c)$,

$$r_n(x) = \int_D \frac{c_d \varphi_n(y)}{|x-y|^{d+1}}\, dy. \tag{3.1}$$

Proof. For any $t > 0$ and $x \in \operatorname{int}(D^c)$, we have

$$\frac{u_n(x,t)}{t} = \frac{P_t \varphi_n(x)}{t} = \int_D \frac{c_d \varphi_n(y)}{(t^2 + |x-y|^2)^{\frac{d+1}{2}}}\, dy.$$

Note that for $x \in \operatorname{int}(D^c)$ we have

$$(t^2 + |x-y|^2)^{-(d+1)/2} \leqslant \delta_D^{-d-1}(x).$$

The bounded convergence theorem implies (3.1). $\square$

Let us recall that if $D \subset \mathbb{R}^d$ is a bounded Lipschitz domain then, by (2.6) and (2.7), there exist $\beta = \beta(D) \in (0,1)$ and a constant $c(D,\beta)$ such that for all $n \in \mathbb{N}$, $x \in D$ we have

$$|\varphi_n(x)| \leqslant c(D,n,\beta)\delta_D^{\beta}(x). \tag{3.2}$$

Proposition 3.2. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $\beta \in (0,1)$ be the constant in* (3.2). *Then*

(i) $$|r_n(x)| \leqslant \int_D \frac{c_d|\varphi_n(y)|}{|x-y|^{d+1}}\,dy \leqslant c(D,n,\beta)\min(\delta_D^{\beta-1}(x), \delta_D^{-d-1}(x)),$$

for all $x \in \operatorname{int}(D^c)$.

(ii) $$h^{-1}|P_h\varphi_n(x) - \varphi_n(x)| \leqslant c(D,n,\beta)\delta_D^{\beta-1}(x),$$

for $x \in D$, $h>0$, *and*

(iii) $$\frac{\partial u_n}{\partial t}(x,t) = -\lambda_n u_n(x,t) + P_t r_n(x), \tag{3.3}$$

for $x \in \mathbb{R}^d$, $t>0$.

Proof. (i) Let $x \in \operatorname{int}(D^c)$. The upper bound $c(D,n,\beta)\delta_D^{-d-1}(x)$ is easy. We have

$$\int_D c_d|\varphi_n(y)||x-y|^{-d-1}\,dy \leqslant c_d\delta_D^{-d-1}(x)\int_D |\varphi_n(y)|\,dy \leqslant c(D,n)\delta_D^{-d-1}(x).$$

On the other hand, by (3.2) we get

$$\int_D c_d|\varphi_n(y)||x-y|^{-d-1}\,dy \leqslant c(D,n,\beta)\int_D \delta_D^{\beta}(y)|x-y|^{-d-1}\,dy. \tag{3.4}$$

We will divide the integral over D into two integrals, one over the set $D \cap B(x, 2\delta_D(x))$ and the other one over the set $D \backslash B(x, 2\delta_D(x))$. Note that for $y \in D \cap B(x, 2\delta_D(x))$ we have $\delta_D(y) \leqslant \delta_D(x)$ and for $y \in D$ we have $\delta_D(y) \leqslant |x-y|$. Hence the integral on the right-hand side of (3.4) is bounded above by

$$\begin{aligned}&\int_{D \cap B(x,2\delta_D(x))} \delta_D^{\beta}(x)|x-y|^{-d-1}\,dy + \int_{D\backslash B(x,2\delta_D(x))} |x-y|^{-d-1+\beta}\,dy\\ &\quad\leqslant \delta_D^{\beta}(x)\int_{B(x,2\delta_D(x))\backslash B(x,\delta_D(x))} |x-y|^{-d-1}\,dy + \int_{B^c(x,2\delta_D(x))} |x-y|^{-d-1+\beta}\,dy.\end{aligned}$$

A simple integration in polar coordinates shows that the sum is dominated above by $c(d,\beta)\delta_D^{\beta-1}(x)$.

(ii) Let $x \in D$ and $h > 0$. As in the proof of Theorem 1.1, we obtain that $h^{-1}|P_h\varphi_n(x) - \varphi_n(x)|$ is bounded above by

$$h^{-1}|P_h^D\varphi_n(x) - \varphi_n(x)| + h^{-1}|P_h\varphi_n(x) - P_h^D\varphi_n(x)|$$
$$\leqslant h^{-1}|e^{-\lambda_n h} - 1||\varphi_n(x)| + h^{-1}\int_D r_D(h,x,y)|\varphi_n(y)|\,dy.$$

The first term in the sum is controlled by $||\varphi_n||_\infty \lambda_n$. It remains to show that

$$h^{-1}\int_D r_D(h,x,y)|\varphi_n(y)|\,dy \leqslant c(D,n,\beta)\delta_D^{\beta-1}(x). \tag{3.5}$$

As in the previous argument, we divide the integral in (3.5) as an integral over $B(x,\delta_D(x))$ and an integral over $D \backslash B(x,\delta_D(x))$. For $y \in B(x,\delta_D(x))$ we use (2.9) to get $h^{-1}r_D(h,x,y) \leqslant c_d\delta_D^{-d-1}(x)$. For $y \in D\backslash B(x,\delta_D(x))$ we estimate $h^{-1}r_D(h,x,y) \leqslant p(h,x,y) \leqslant c_d|x-y|^{-d-1}$. Note that $2|y-x| \geqslant \delta_D(y)$ for $y \in D\backslash B(x,\delta_D(x))$. Applying (3.2) we obtain that $|\varphi_n(y)| \leqslant c(D,n,\beta)(2\delta_D^\beta(x))^\beta$ for $y \in B(x,\delta_D(x))$ and $|\varphi_n(y)| \leqslant c(D,n,\beta)\delta_D^\beta(y) \leqslant c(D,n,\beta)(2|x-y|)^\beta$, for $y \in D\backslash B(x,\delta_D(x))$. It follows that

$$\begin{aligned} h^{-1}\int_D r_D(h,x,y)|\varphi_n(y)|\,dy \leqslant\ & c(D,n,\beta)\delta_D^{-d-1+\beta}(x)\int_{B(x,\delta_D(x))} dy \\ & + c(D,n,\beta)\int_{D\backslash B(x,\delta_D(x))} |x-y|^{-d-1+\beta}\,dy \\ \leqslant\ & c(D,n,\beta)\delta_D^{\beta-1}(x), \end{aligned}$$

which proves (3.5) and (ii).

(iii) Let $t > 0$ and $x \in \mathbb{R}^d$.

$$\frac{\partial u_n}{\partial t}(x,t) = \lim_{h\to 0^+}\int p(t,x,y)\frac{P_h\varphi_n(y) - \varphi_n(y)}{h}\,dy$$
$$= \lim_{h\to 0^+}\int_{\mathrm{int}(D^c)} p(t,x,y)\frac{P_h\varphi_n(y) - \varphi_n(y)}{h}\,dy \tag{3.6}$$
$$+ \lim_{h\to 0^+}\int_D p(t,x,y)\frac{P_h\varphi_n(y) - \varphi_n(y)}{h}\,dy. \tag{3.7}$$

If $y \in \mathrm{int}(D^c)$, then $h^{-1}(P_h\varphi_n(y) - \varphi_n(y)) = h^{-1}P_h\varphi_n(y)$ tends to $r_n(y)$ when $h \to 0^+$. If $y \in D$ then by Theorem 1.1 $h^{-1}(P_h\varphi_n(y) - \varphi_n(y))$ tends to $-\lambda_n\varphi_n(y)$ when $h \to 0^+$. Hence, to show (3.3) it remains to justify the change of the limit and the integral in

(3.6) and (3.7). For $x \in \operatorname{int}(D^c)$ we have

$$\left|\frac{P_h\varphi_n(x) - \varphi_n(x)}{h}\right| = \left|\frac{P_h\varphi_n(x)}{h}\right| \leqslant \int_D \frac{c_d|\varphi_n(y)|}{|x-y|^{d+1}}\,dy.$$

By (i) this is bounded above by $c(D,n,\beta)\min(\delta_D^{\beta-1}(x), \delta_D^{-d-1}(x))$, where $x \in \operatorname{int}(D^c)$ which is an integrable function on $\operatorname{int}(D^c)$. Similarly (ii) shows that $h^{-1}(P_h\varphi_n(y) - \varphi_n(y))$ for $y \in D$ is bounded above by the function $c(D,n,\beta)\delta_D^{\beta-1}(x)$ which is integrable on D. Therefore by the bounded convergence theorem we can change limits and integrals in (3.6) and (3.7). This proves (3.3) and completes the proof of the proposition. □

Our aim now is to obtain a variational formulas for the eigenvalues λ_n of the following type:

$$\lambda_n = \inf_{u \in \mathscr{F}_n} \int_H |\nabla u(x,t)|^2\,dx\,dt,$$

for a suitably chosen class of function $\mathscr{F}_n$. For a function $u : H \to \mathbb{R}$ we denote $\nabla u = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d}, \frac{\partial u}{\partial t})$. For $\varepsilon > 0$ we put $H_\varepsilon = \{(x,t) : x \in \mathbb{R}^d,\ t > \varepsilon\}$. Recall that $H_+ = \{(x,t) : t > 0,\ x \in \mathbb{R}^d\}$. We need some estimates on ∇u_n. These are obtained essentially by differentiating under the integral sign using our representation for the functions $u_n(x,t)$ in terms of the Cauchy (Poisson) kernel. These calculations are in fact very similar to those in [46, Chapter IV].

Lemma 3.3. *Let $D \subset \mathbb{R}^d$ be a bounded domain. For any $\varepsilon > 0$, $(x,t) \in H_+$ and $i = 1, \dots, d$, we have*

(a)
$$\frac{\partial u_n}{\partial x_i}(x,t) = -c_d \int_D \frac{(d+1)t(x_i - y_i)}{(t^2 + |x-y|^2)^{\frac{d+3}{2}}}\varphi_n(y)\,dy,$$

(b)
$$\frac{\partial u_n}{\partial t}(x,t) = c_d \int_D \frac{|x-y|^2 - dt^2}{(t^2 + |x-y|^2)^{\frac{d+3}{2}}}\varphi_n(y)\,dy,$$

(c)
$$\frac{\partial^2 u_n}{\partial x_i^2}(x,t) = c_d \int_D \frac{-(d+1)t(t^2 + |x-y|^2) + (d+1)(d+3)t(x_i - y_i)^2}{(t^2 + |x-y|^2)^{\frac{d+5}{2}}}\varphi_n(y)\,dy,$$

and

(d)
$$\frac{\partial^2 u_n}{\partial t^2}(x,t) = c_d \int_D \frac{-3(d+1)t|x-y|^2 + d(d+1)t^3}{(t^2+|x-y|^2)^{\frac{d+5}{2}}} \varphi_n(y)\, dy.$$

For any $\varepsilon > 0$ *there exists a constant* $c = c(D, n, \varepsilon)$ *such that for all* $(x,t) \in H_\varepsilon$ *we have*

(e)
$$|\nabla u_n(x,t)| \leqslant c(t^2+|x|^2)^{-(d+1)/2},$$

and

(f)
$$\left|\frac{\partial^2 u_n}{\partial t^2}(x,t)\right| + \sum_{i=1}^d \left|\frac{\partial^2 u_n}{\partial x_i^2}(x,t)\right| \leqslant \frac{c}{(t^2+|x|^2)^{\frac{d+2}{2}}}.$$

In particular,

(g)
$$\int_{H_\varepsilon} |\nabla u_n(x,t)|^2\, dx\, dt < \infty$$

and

(h)
$$\int_{H_\varepsilon} \left|\frac{\partial^2 u_n}{\partial t^2}(x,t)\right| + \sum_{i=1}^d \left|\frac{\partial^2 u_n}{\partial x_i^2}(x,t)\right| dx\, dt < \infty.$$

Proof. Recall that $u_n(x,t) = \int_D p(t,x,y)\varphi_n(y)\, dy$. Formulas (a)–(d) follow from our explicit expression for $p(t,x,y)$ in (1.1) and the fact that

$$\frac{\partial u_n}{\partial x_i}(x,t) = \int_D \frac{\partial}{\partial x_i} p(t,x,y)\varphi_n(y),\ dy \quad (x,t) \in H_+,$$

with a similar formula for $\frac{\partial u_n}{\partial t}(x,t)$.

Let $a = 2\,\mathrm{diam}(D) + 2\,\mathrm{dist}(0,D)$. For $y \in D$ we have $|y| \leqslant a/2$. So, for $|x| \geqslant a$ and $y \in D$ we get

$$|x-y| \geqslant |x| - |y| \geqslant |x| - a/2 \geqslant |x|/2$$

and

$$|x-y| \leqslant |x| + |y| \leqslant |x| + a/2 \leqslant 3|x|/2.$$

For $y \in D$ and $|x| < a$ we have $|x-y| \leqslant 3a/2$. We will use these elementary observations several times below.

For $t > \varepsilon$ and $|x| \geqslant a$,

$$\left|\frac{\partial u_n}{\partial x_i}(x,t)\right| \leqslant c_d(d+1)\int_D \frac{(3/2)t|x|}{(t^2+|x|^2/4)^{\frac{d+3}{2}}}|\varphi_n(y)|\,dy$$
$$\leqslant \frac{c(d)}{(t^2+|x|^2)^{\frac{d+1}{2}}}\int_D |\varphi_n(y)|\,dy.$$

For $t > \varepsilon$ and $|x| < a$,

$$\left|\frac{\partial u_n}{\partial x_i}(x,t)\right| \leqslant c_d(d+1)\int_D \frac{(3/2)ta}{t^{d+3}}|\varphi_n(y)|\,dy$$
$$\leqslant \frac{c(D,n)}{t^{d+2}}\frac{t}{\varepsilon} \leqslant \frac{c(D,n)}{t^{d+1}}.$$

But for $t > \varepsilon$ and $|x| < a$, $|x| < ta/\varepsilon$ so

$$\frac{1}{t^{d+1}} \leqslant \frac{1}{(t^2+|x|^2)^{\frac{d+1}{2}}}\left(1+\frac{a^2}{\varepsilon^2}\right)^{\frac{d+1}{2}}. \tag{3.8}$$

Therefore, for t, x as above,

$$\left|\frac{\partial u_n}{\partial x_i}(x,t)\right| \leqslant \frac{c(D,n)}{(t^2+|x|^2)^{\frac{d+1}{2}}}\left(1+\frac{a^2}{\varepsilon^2}\right)^{\frac{d+1}{2}}.$$

Similarly, for $t > \varepsilon$ and $|x| \geqslant a$,

$$\left|\frac{\partial u_n}{\partial t}(x,t)\right| \leqslant c_d\int_D \frac{(dt^2+(9/4)|x|^2)}{(t^2+|x|^2/4)^{\frac{d+3}{2}}}|\varphi_n(y)|\,dy$$
$$\leqslant \frac{c(d)}{(t^2+|x|^2)^{\frac{d+1}{2}}}\int_D |\varphi_n(y)|\,dy.$$

On the other hand, for $t > \varepsilon$ and $|x| < a$,

$$\left|\frac{\partial u_n}{\partial t}(x,t)\right| \leqslant c_d\int_D \frac{(dt^2+(9/4)a^2)}{t^{d+3}}|\varphi_n(y)|\,dy$$
$$\leqslant \frac{c(D,n,\varepsilon)}{t^{d+1}} \leqslant \frac{c(D,n,\varepsilon)}{(t^2+|x|^2)^{\frac{d+1}{2}}}.$$

The last two inequalities follow from the fact that $a^2 \leqslant t^2a^2/\varepsilon^2$ and (3.8). Now, (e) follows from the above inequalities.

Estimate (g) is a simple consequence of (e). In fact,

$$\int_{H_\varepsilon} |\nabla u_n(x,t)|^2 \, dx\, dt \leqslant c(D,n,\varepsilon) \int_{H_\varepsilon} \frac{dx\, dt}{(t^2+|x|^2)^{d+1}} < \infty .$$

We will now prove (f). For $t>\varepsilon$ and $|x|\geqslant a$,

$$\begin{aligned}\left|\frac{\partial^2 u_n}{\partial x_i^2}(x,t)\right| &\leqslant c(d)\int_D \frac{(t^2+(9/4)|x|^2)^{3/2}}{(t^2+|x|^2/4)^{\frac{d+5}{2}}}\,|\varphi_n(y)|\,dy \\ &\leqslant \frac{c(D,n)}{(t^2+|x|^2)^{\frac{d+2}{2}}}\end{aligned}$$

and for $t>\varepsilon$ and $|x|<a$,

$$\left|\frac{\partial^2 u_n}{\partial x_i^2}(x,t)\right| \leqslant c(d)\int_D \frac{(t^2+(9/4)a^2)^{3/2}}{t^{d+5}}\,|\varphi_n(y)|\,dy \leqslant \frac{c(D,n,\varepsilon)}{t^{d+2}}.$$

But for $t>\varepsilon$ and $|x|<a$,

$$\frac{1}{t^{d+2}} \leqslant \frac{1}{(t^2+|x|^2)^{\frac{d+2}{2}}}\left(1+\frac{a^2}{\varepsilon^2}\right)^{\frac{d+2}{2}}.$$

Therefore, for t and x as above,

$$\left|\frac{\partial^2 u_n}{\partial x_i^2}(x,t)\right| \leqslant \frac{c(D,n,\varepsilon)}{(t^2+|x|^2)^{\frac{d+2}{2}}}.$$

Similarly, for $t>\varepsilon$ and $|x|\geqslant a$,

$$\left|\frac{\partial^2 u_n}{\partial t^2}(x,t)\right| \leqslant \int_D \frac{c(d)(t^2+(9/4)|x|^2)t}{(t^2+|x|^2/4)^{\frac{d+5}{2}}}|\varphi_n(y)|\,dy \leqslant \frac{c(D,n)}{(t^2+|x|^2)^{\frac{d+2}{2}}}$$

and for $t>\varepsilon$ and $|x|<a$,

$$\left|\frac{\partial^2 u_n}{\partial t^2}(x,t)\right| \leqslant \int_D \frac{c(d)(t^2+(9/4)a^2)t}{t^{d+5}}\,|\varphi_n(y)|\,dy \leqslant \frac{c(D,n,\varepsilon)}{(t^2+|x|^2)^{\frac{d+2}{2}}}.$$

The previous two inequalities imply (f). Finally, (h) follows from this. $\square$

We will now introduce the class of function $\mathscr{F}(D)=\mathscr{F}$ which we shall use in the variational characterization of λ_n. Motivated by Lemma 3.3, we define this class as follows.

Definition 3.1. Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. We define $\mathscr{F}(D)$ to be the collection of all finite linear combinations of functions $u : H \to \mathbb{R}$ satisfying the following conditions:

(i) u is continuous on H except possibly on $\{(x,0) : x \in \partial D\}$ and u is bounded on H.

(ii) $\nabla u(x,t)$ exists for almost all $(x,t) \in H_+$ and ∇u is a measurable function. If $(x,t) \in H_+$ and $\nabla u(x,t)$ does not exists then $u(x,t) = 0$. Moreover, for all $\varepsilon > 0$ there exists a constant $c(\varepsilon)$ such that for all $t > \varepsilon$,

$$|\nabla u(x,t)| \leqslant c(\varepsilon)(t^2 + |x|^2)^{-(d+1)/2},$$

for any $(x,t) \in H$ for which $\nabla u(x,t)$ exists.

(iii) $u(x,0) = 0$ for $x \in D^c \backslash \partial D$ and

$$\int_{\mathbb{R}^d} u^2(x,0)\, dx < \infty.$$

(iv)
$$\int_H |\nabla u(x,t)|^2\, dx\, dt < \infty.$$

The space $\mathscr{F}(D)$ is the linear space spanned by the functions $u : H \to \mathbb{R}$ which satisfy (i)–(iv). We will often simply write $\mathscr{F}$ for $\mathscr{F}(D)$ unless we want to stress the dependence on D. Notice that the condition "if $\nabla u(x,t)$ does not exists, then $u(x,t) = 0$," is the only condition which prevents the class of functions $u : H \to \mathbb{R}$ satisfying (i)–(iv) from being a linear space itself. This condition, as it turns out, will be very important in the sequel. Finally, we note that $u_n(x,t)$ satisfies (i)–(iii) by Lemma 3.3 and via the Fourier transform one can easily show that it also satisfies (iv). An alternative way to verify this which has some additional advantages, as we shall see below, is to use Green's theorem. In particular, we need to justify the use of Green's formula on expressions of the form

$$\int_H \nabla u(x,t) \nabla u_n(x,t)\, dx\, dt,$$

for $u \in \mathscr{F}$. Some of the "Littlewood–Paley" formulas below can also be derived by the Fourier transform. We choose to prove them by integration by parts since the Fourier transform method does not suffice for all our formulas.

Proposition 3.4. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. If $u : H \to \mathbb{R}$ satisfies conditions* (i)–(iii) *in Definition* 3.1 *then for any $\varepsilon > 0$ and $n \in \mathbb{N}$ we have*

$$\int_{H_\varepsilon} \nabla u(x,t) \nabla u_n(x,t)\, dx\, dt = -\int_{\mathbb{R}^d} u(x,\varepsilon) \frac{\partial u_n}{\partial t}(x,\varepsilon)\, dx. \tag{3.9}$$

In particular, both integrals finite.

We interpret (3.9) as saying that Green's formula can be applied to

$$\int_{H_\varepsilon} \nabla u(x,t)\nabla u_n(x,t)\,dx\,dt$$

in that

$$\begin{aligned}\int_{H_\varepsilon} \nabla u(x,t)\nabla u_n(x,t)\,dx\,dt = &-\int_{H_\varepsilon} u(x,t)\Delta u_n(x,t)\,dx\,dt\\ &-\int_{\mathbb{R}^d} u(x,\varepsilon)\,\frac{\partial u_n}{\partial t}(x,\varepsilon)\,dx.\end{aligned}$$

The $(-)$ sign rather than the $(+)$ sign arises because we are using $\frac{\partial}{\partial t}$ for the inner normal derivative at ∂H_ε.

Proof. By the Lemma 3.3 and (ii) in Definition 3.1

$$\int_{H_\varepsilon} |\nabla u(x,t)||\nabla u_n(x,t)|\,dx\,dt \leqslant c(\varepsilon,D,n,u)\int_{H_\varepsilon}(t^2+|x|^2)^{-(d+1)}\,dx\,dt < \infty.$$

So the integral

$$\int_{H_\varepsilon} \nabla u(x,t)\nabla u_n(x,t)\,dx\,dt$$

is absolutely convergent. Moreover, this integral equals

$$\begin{aligned}&\sum_{i=1}^d \int_\varepsilon^\infty \underbrace{\int_{-\infty}^\infty \cdots \int_{-\infty}^\infty}_{(d-1)\text{-integrals}} \int_{-\infty}^\infty \frac{\partial u}{\partial x_i}(x,t)\,\frac{\partial u_n}{\partial x_i}(x,t)\,dx_i \underbrace{dx_1\ldots dx_d}_{(d-1)\ \text{times without}\ dx_i}\,dt\\ &+\int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \int_\varepsilon^\infty \frac{\partial u}{\partial t}(x,t)\,\frac{\partial u_n}{\partial t}(x,t)\,dt\,dx_1\ldots dx_d.\end{aligned}$$

By Lemma 3.3 and (ii) in Definition 3.1 we get for each $t \geqslant \varepsilon$,

$$\begin{aligned}\int_{-\infty}^\infty \left|\frac{\partial u}{\partial x_i}(x,t)\,\frac{\partial u_n}{\partial x_i}(x,t)\right| dx_i &\leqslant \int_{-\infty}^\infty \frac{c(D,\varepsilon,n,u)\,dx_i}{(t^2+|x|^2)^{d+1}}\\ &\leqslant \int_{-\infty}^\infty \frac{c(D,\varepsilon,n,u)\,dx_i}{(\varepsilon^2+|x|^2)^{d+1}} < \infty.\end{aligned}$$

We now claim that

$$\int_{-\infty}^\infty \frac{\partial u}{\partial x_i}(x,t)\,\frac{\partial u_n}{\partial x_i}(x,t)\,dx_i = -\int_{-\infty}^\infty u(x,t)\,\frac{\partial u_n}{\partial x_i^2}(x,t)\,dx_i. \tag{3.10}$$

We may assume $i = 1$. Let us fix the coordinates $x_2, \dots, x_d, t$ and put $\Omega = \{x_1 \in (-\infty, \infty) : u(x,t) = u(x_1, x_2, \dots, x_d, t) = 0\}$. The set $(-\infty, \infty) \backslash \Omega$ consists of at most countably many intervals $(a_k, b_k)_{k=1}^{\infty}$ such that for all $x_1 \in (a_k, b_k)$ we have $u(x,t) \neq 0$ (some of intervals (a_k, b_k) may be unbounded). For $x_1 = a_k$ or $x_1 = b_k$ (when $a_k \neq -\infty$, $b_k \neq \infty$) we have $u(x,t) = 0$. Hence

$$\int_{-\infty}^{\infty} \frac{\partial u}{\partial x_1}(x,t) \frac{\partial u_n}{\partial x_1}(x,t)\, dx_1 = \sum_{k=1}^{\infty} \int_{a_k}^{b_k} \frac{\partial u}{\partial x_1}(x,t) \frac{\partial u_n}{\partial x_1}(x,t)\, dx_1. \tag{3.11}$$

On the other hand,

$$\int_{a_k}^{b_k} \frac{\partial u}{\partial x_1}(x,t) \frac{\partial u_n}{\partial x_1}(x,t)\, dx_1 = \left[u(x,t) \frac{\partial u_n}{\partial x_1}(x,t) \right]_{x_1 = a_k}^{x_1 = b_k} - \int_{a_k}^{b_k} u(x,t) \frac{\partial^2 u_n}{\partial x_1^2}(x,t)\, dx_1. \tag{3.12}$$

If $a_k = -\infty$, then the expression

$$\left[u(x,t) \frac{\partial u_n}{\partial x_1}(x,t) \right]_{x_1 = a_k}$$

should be understood in the limit sense. By (i) in Definition 3.1 and (e) of Lemma 3.3, we get for $a_k = -\infty$ that this expression is equal to 0. Similarly, if $b_k = \infty$, then

$$\left[u(x,t) \frac{\partial u_n}{\partial x_1}(x,t) \right]^{x_1 = b_k} = 0.$$

When $a_k \neq -\infty$ and $b_k \neq \infty$, we have $u(a_k, x_2, \dots, x_d, t) = 0$ and we have $u(b_k, x_2, \dots, x_d, t) = 0$. Hence,

$$\left[u(x,t) \frac{\partial u_n}{\partial x_1}(x,t) \right]_{x_1 = a_k} = \left[u(x,t) \frac{\partial u_n}{\partial x_1}(x,t) \right]^{x_1 = b_k} = 0.$$

Note also that by (i) in Definition 3.1 and by (f) in Lemma 3.3, we have

$$\int_{-\infty}^{\infty} |u(x,t)| \left| \frac{\partial^2 u_n}{\partial x_1^2}(x,t) \right| dx_1 < \infty$$

and it follows that

$$\int_{-\infty}^{\infty} u(x,t) \frac{\partial^2 u_n}{\partial x_1^2}(x,t)\, dx_1$$

is absolutely convergent. Therefore (3.11) and (3.12) give (3.10). By similar arguments for any $x \in \mathbb{R}^d$,

$$\begin{aligned}\int_{\varepsilon}^{\infty} \frac{\partial u}{\partial t}(x,t) \frac{\partial u_n}{\partial t}(x,t)\, dt \\ = \left[u(x,t) \frac{\partial u_n}{\partial t}(x,t) \right]_{t=\varepsilon}^{t=\infty} - \int_{\varepsilon}^{\infty} u(x,t) \frac{\partial^2 u_n}{\partial t^2}(x,t)\, dt.\end{aligned}$$

Both integrals are well defined for each $x \in \mathbb{R}^d$. As before, the expression

$$\left[u(x,t) \frac{\partial u_n}{\partial t}(x,t) \right]^{t=\infty}$$

should be understood as a limit and it equals 0. By repeated integration, it follows that the left-hand side of (3.9) is

$$\begin{aligned}&- \sum_{i=1}^{d} \int_{\varepsilon}^{\infty} \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{d \text{ integrals}} u(x,t) \frac{\partial^2 u_n}{\partial x_i^2}(x,t)\, dx_i \underbrace{dx_1, \ldots, dx_d}_{(d-1) \text{ times without } dx_i} dt \\ &- \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u(x,\varepsilon) \frac{\partial u_n}{\partial t}(x,\varepsilon)\, dx_1, \ldots, dx_d \\ &- \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{\varepsilon}^{\infty} u(x,t) \frac{\partial^2 u_n}{\partial t^2}(x,t)\, dt\, dx_1, \ldots, dx_d.\end{aligned}$$

By Fubini theorem this is

$$- \int_{H_\varepsilon} u(x,t) \Delta u_n(x,t)\, dx\, dt - \int_{\mathbb{R}^d} u(x,\varepsilon) \frac{\partial u_n}{\partial t}(x,\varepsilon)\, dx.$$

Since $\Delta u_n(x,t) = 0$, we obtain (3.9). The use of Fubini theorem is justified by the following observation. By (i) in Definition 3.1 and by (h) and (e) in Lemma 3.3, we have

$$\int_{H_\varepsilon} |u(x,\varepsilon)| \left(\left| \frac{\partial^2 u_n}{\partial t^2}(x,t) \right| + \sum_{i=1}^{d} \left| \frac{\partial^2 u_n}{\partial x_1^2}(x,t) \right| \right) dx\, dt < \infty$$

and

$$\int_{\mathbb{R}^d} |u(x,\varepsilon)| \left| \frac{\partial u_n}{\partial t}(x,\varepsilon) \right| dx \leqslant \int_{\mathbb{R}^d} \frac{||u||_\infty c(D,n,\varepsilon)\, dx}{(\varepsilon^2 + |x|^2)^{(d+1)/2}} < \infty.$$

This also shows that the right-hand side of (3.9) is an absolutely convergent integral. $\square$

Proposition 3.5. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. If $u: H \to \mathbb{R}$ satisfies conditions* (i)–(iii) *of Definition* 3.1, *then for any $n \in \mathbb{N}$ we have*

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d} u(x,\varepsilon) u_n(x,\varepsilon)\, dx = \int_D u(x,0) \varphi_n(x)\, dx \tag{3.13}$$

and

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d} |u(x,\varepsilon)||P_\varepsilon r_n(x)|\, dx = 0. \tag{3.14}$$

In particular,

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d} u(x,\varepsilon) \frac{\partial u_n}{\partial t}(x,\varepsilon)\, dx = -\lambda_n \int_D u(x,0) \varphi_n(x)\, dx. \tag{3.15}$$

Proof. Note that $u_n(x,\varepsilon) \to \varphi_n(x)$ as $\varepsilon \to 0^+$ for any $x \in \mathbb{R}^d$ (recall that φ_n is extended to the whole of $\mathbb{R}^d$). Similarly, by (i) in Definition 3.1 $u(x,\varepsilon) \to u(x,0)$ as $\varepsilon \to 0^+$ for all $x \in \mathbb{R}^d \backslash \partial D$. Hence this limit exists for almost all $x \in \mathbb{R}^d$. By (i) in Definition 3.1 u is bounded and for any $x \in \mathbb{R}^d$ and $\varepsilon \in (0,1)$ we have

$$|u_n(x,\varepsilon)| = |P_\varepsilon \varphi_n(x)| \leqslant c(D)||\varphi_n||_\infty (1 + \delta_D(x))^{-d-1}.$$

Hence (3.13) follows by the bounded convergence theorem.

The proof of (3.14) is more complicated. By (2.6) and (3.1) we have $|r_n(x)| \leqslant c(n,D) r_1(x)$, so to prove (3.14) it is sufficient to show that

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d} |u(x,\varepsilon)| P_\varepsilon r_1(x)\, dx = 0. \tag{3.16}$$

Fix $h > 0$ and put $D_h = \{x \in \mathbb{R}^d : \operatorname{dist}(x,D) \leqslant h\}$. For $x \in \mathbb{R}^d$, define

$$f_h(x) = r_1(x) 1_{D_h}(x) \quad \text{and} \quad \tilde{f}_h(x) = r_1(x) 1_{D_h^c}(x)$$

so that $P_\varepsilon r_1 = P_\varepsilon f_h + P_\varepsilon \tilde{f}_h$. Clearly, for any $h, \varepsilon > 0$ we have

$$\begin{aligned}\int_{\mathbb{R}^d} |u(x,\varepsilon)| P_\varepsilon f_h(x)\, dx &\leqslant ||u||_\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p(\varepsilon,x,y) f_h(y)\, dy\, dx \\ &= ||u||_\infty \int_{\mathbb{R}^d} f_h(y) \int_{\mathbb{R}^d} p(\varepsilon,x,y)\, dx\, dy = ||u||_\infty ||f_h||_1. \end{aligned} \tag{3.17}$$

By (i) in Proposition 3.2 we see that $r_1 \in L^1(\mathbb{R}^d)$. Also, $\operatorname{supp}(r_1) \subset D^c$. It follows that $\lim_{h \to 0^+} ||f_h||_1 = 0$. Hence for any $\varepsilon > 0$ the left-hand side of (3.17) tends to 0 as $h \to 0^+$. Thus by choosing a sufficiently small $h > 0$, the integral on the left-hand side of (3.17) can be made arbitrarily small independently of our choice of $\varepsilon > 0$. Therefore to prove (3.16) it suffices to show that for each fixed $h > 0$ we have

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d} |u(x,\varepsilon)| P_\varepsilon \tilde{f}_h(x)\, dx = 0. \tag{3.18}$$

By Proposition 3.2 there exists $\beta = \beta(D) \in (0,1)$ such that for all $x \in \operatorname{int}(D^c)$,

$$\tilde{f}_h(x) = r_1(x) 1_{D_h^c}(x) \leqslant c(D,\beta) \min(\delta_D^{\beta-1}(x), \delta_D^{-d-1}(x)).$$

It follows that $\tilde{f}_h \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.

Let us denote by $D' = \{x \in D^c : \delta_D(x) \leqslant \operatorname{diam}(D)\}$ and $D'' = \{x \in D^c : \delta_D(x) > \operatorname{diam}(D)\}$. Let $x \in D$. For any $y \in D_h^c$ we have $|x-y| \geqslant h$ and $|x-y| \geqslant \delta_D(y)$. It follows that

$$\begin{aligned} P_\varepsilon \tilde{f}_h(x) &= \int_{D_h^c} \frac{c_d \varepsilon}{(\varepsilon^2 + |x-y|^2)^{\frac{d+1}{2}}} \tilde{f}_h(y)\, dy \\ &\leqslant \varepsilon c_d ||\tilde{f}_h||_\infty \int_{D_h^c \cap D''} \frac{dy}{(\delta_D(y))^{d+1}} + \varepsilon c_d ||\tilde{f}_h||_\infty \int_{D_h^c \cap (D \cup D')} \frac{dy}{h^{d+1}}. \end{aligned}$$

Hence $P_\varepsilon \tilde{f}_h(x) \leqslant c(D,h)\varepsilon$ for $x \in D$. Obviously,

$$\int_D |u(x,\varepsilon)| P_\varepsilon \tilde{f}_h(x)\, dx \leqslant ||u||_\infty \int_D P_\varepsilon \tilde{f}_h(x)\, dx.$$

Therefore by the bounded convergence theorem,

$$\lim_{\varepsilon \to 0^+} \int_D |u(x,\varepsilon)| P_\varepsilon \tilde{f}_h(x)\, dx = 0. \tag{3.19}$$

Let $x \in D' \backslash \partial D$. Then $\lim_{\varepsilon \to 0^+} u(x,\varepsilon) = 0$ and $P_\varepsilon \tilde{f}_h(x) \leqslant ||\tilde{f}_h||_\infty$. Recall also that u is bounded and the Lebesgue measure of ∂D is zero. Hence by the bounded convergence theorem again,

$$\lim_{\varepsilon \to 0^+} \int_{D'} |u(x,\varepsilon)| P_\varepsilon \tilde{f}_h(x)\, dx = 0. \tag{3.20}$$

Now, let $x \in D''$. Note that for $y \in B(x, \delta_D(x)/2)$ we have $\delta_D(y) \geqslant \delta_D(x)/2$ and

$$\tilde{f}_h(y) \leqslant r_1(y) \leqslant c(D,\beta) \delta_D^{-d-1}(y) \leqslant c' \delta_D^{-d-1}(x),$$

where $c' = 2^{d+1}c(D,\beta)$. For $y \notin B(x, \delta_D(x)/2)$ we have $|x-y| \geqslant \delta_D(x)/2$. Therefore,

$$\begin{aligned}
P_\varepsilon \tilde{f}_h(x) &= \int_{D_h^c} \frac{c_d \varepsilon}{(\varepsilon^2 + |x-y|^2)^{\frac{d+1}{2}}} \tilde{f}_h(y)\, dy \\
&\leqslant \int_{B(x,\delta_D(x)/2)} \frac{c_d \varepsilon c'\, dy}{(\varepsilon^2 + |x-y|^2)^{\frac{d+1}{2}} \delta_D^{d+1}(x)} + \int_{B^c(x,\delta_D(x)/2)} \frac{c_d \varepsilon \tilde{f}_h(y)\, dy}{(\delta_D(x)/2)^{d+1}}.
\end{aligned}$$

The first integral is bounded by $c'\varepsilon\delta_D^{-d-1}(x)$ and the second one is bounded by $\varepsilon c(D)\delta_D^{-d-1}(x)||\tilde{f}_h||_1$. The function $\delta_D^{-d-1}(x)$ is integrable on D''. Finally, for $x \in D''$ we also have $\lim_{\varepsilon \to 0^+} u(x,\varepsilon) = 0$. Hence, by bounded convergence theorem,

$$\lim_{\varepsilon \to 0^+} \int_{D''} |u(x,\varepsilon)| P_\varepsilon \tilde{f}_h(x)\, dx = 0.$$

This together with (3.19) and (3.18) gives (3.14). We have proved (3.14). Finally, (3.15) follows from (3.13), (3.14) and (3.3). $\square$

Proposition 3.6. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then*

$$\int_H |\nabla u_n(x,t)|^2\, dx\, dt = \lambda_n, \quad n \in \mathbb{N}.$$

In particular, we conclude that u_n satisfies (iv) *of Definition* 3.1 *and hence $u_n \in \mathscr{F}$.*

Proof. Since u_n satisfies conditions (i)–(iii) we can apply (3.9) and (3.15). This gives

$$\begin{aligned}
\int_H |\nabla u_n(x,t)|^2\, dx\, dt &= \lim_{\varepsilon \to 0^+} \int_{H_\varepsilon} |\nabla u_n(x,t)|^2\, dx\, dt \\
&= -\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d} u_n(x,\varepsilon) \frac{\partial u_n}{\partial t}(x,\varepsilon)\, dx \\
&= \lambda_n \int_D u_n(x,0)\varphi_n(x)\, dx = \lambda_n. \qquad \square
\end{aligned}$$

Proposition 3.7. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $u \in \mathscr{F}$. Then for any $n \in \mathbb{N}$,*

$$\int_H \nabla u(x,t) \nabla u_n(x,t)\, dx\, dt = \lambda_n \int_D u(x,0)\varphi_n(x)\, dx. \tag{3.21}$$

In particular, both integrals are finite.

We note that (3.21) is the "limiting" case of Proposition 3.4 and we again interpret it as the statement that Green's theorem can be applied in the sense that

$$\int_H \nabla u(x,t)\nabla u_n(x,t)\,dx\,dt = -\int_H u(x,t)\Delta u_n(x,t)\,dx\,dt - \int_{\mathbb{R}^d} u(x,0)\frac{\partial u_n}{\partial t}(x,0)\,dx.$$

Identity (3.21) is a "polarized" version of Lemma 2 [46, p. 87], customized for our purposes.

Proof. Since $u \in \mathscr{F}$, $u(x,t) = \sum_{m=1}^k c_m w_m(x,t)$, where the functions w_m satisfy (i)–(iv) in Definition 3.1 and $c_m \in \mathbb{R}$, $m = 1, \dots, k$. Then

$$\lim_{\varepsilon \to 0^+} \int_{H_\varepsilon} \nabla u(x,t)\nabla u_n(x,t)\,dx\,dt = \sum_{m=1}^k c_m \lim_{\varepsilon \to 0^+} \int_{H_\varepsilon} \nabla w_m(x,t)\nabla u_n(x,t)\,dx\,dt.$$

By Propositions 3.4 and 3.5 this is equal to

$$-\sum_{m=1}^k c_m \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d} w_m(x,\varepsilon)\frac{\partial u_n}{\partial t}(x,\varepsilon)\,dx = \sum_{m=1}^k c_m \lambda_n \int_D w_m(x,0)\varphi_n(x)\,dx = \lambda_n \int_D u(x,0)\varphi_n(x)\,dx.$$

Since u_n, u satisfy (iv),

$$\lim_{\varepsilon \to 0^+} \int_{H_\varepsilon} \nabla u(x,t)\nabla u_n(x,t)\,dx\,dt = \int_H \nabla u(x,t)\nabla u_n(x,t)\,dx\,dt,$$

which proves the proposition. □

We now define our "variational" spaces for the Cauchy processes. For any $u : H \to \mathbb{R}$ we put $\tilde{u}(x) = u(x,0)$, $x \in \mathbb{R}^d$ and

$$\|\tilde{u}\|_2 = \left(\int_D \tilde{u}^2(x)\,dx\right)^{1/2}.$$

Let

$$\mathscr{F}_1(D) = \{u \in \mathscr{F}(D) : \|\tilde{u}\|_2 = 1\},$$

and for $n \geqslant 2$, let

$$\mathscr{F}_n(D) = \{u \in \mathscr{F}(D) : \tilde{u} \perp \varphi_1, \dots, \varphi_{n-1};\ \|\tilde{u}\|_2 = 1\}.$$

As before, if there is no danger of confusion, we simply write $\mathcal{F}_n$ for $\mathcal{F}_n(D)$. Our variational formula for λ_n is

Theorem 3.8. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then*

$$\lambda_n = \inf_{u \in \mathcal{F}_n} \int_H |\nabla u(x,t)|^2 \, dx \, dt,$$

for all $n \in \mathbb{N}$.

Proof. To simplify notation, set

$$Q(u,v) = \int_H \nabla u(x,t) \nabla v(x,t) \, dx \, dt. \tag{3.22}$$

We must show that $\lambda_n = \inf_{u \in \mathcal{F}_n} Q(u,u)$. By Proposition 3.6,

$$\inf_{u \in \mathcal{F}_n} Q(u,u) \leqslant Q(u_n,u_n) = \lambda_n.$$

It remains to show that $\inf_{u \in \mathcal{F}_n} Q(u,u) \geqslant \lambda_n$. Fix $u \in \mathcal{F}_n$. For any $k \in \mathbb{N}$ and $(x,t) \in H$, set $v_k(x,t) = \sum_{m=1}^k c_m u_m(x,t)$ where $c_m = \int_D \tilde{u}(x) \varphi_m(x) \, dx$. Since $\mathcal{F}$ is a linear space, $v_k \in \mathcal{F}$. Therefore

$$Q(u,u) = Q(v_k,v_k) + Q(u - v_k, u - v_k) + 2Q(u - v_k, v_k). \tag{3.23}$$

We have

$$Q(u - v_k, v_k) = \sum_{m=1}^k c_m Q(u,u_m) - \sum_{m=1}^k c_m Q(v_k,u_m).$$

By Proposition 3.7 this is equal to

$$\sum_{m=1}^k c_m \lambda_m \int_D u(x,0) \varphi_m(x) \, dx - \sum_{m=1}^k c_m \lambda_m \int_D v_k(x,0) \varphi_m(x) \, dx. \tag{3.24}$$

But

$$\int_D u(x,0) \varphi_m(x) \, dx = c_m$$

and for $m = 1, \ldots, k$,

$$\int_D v_k(x,0) \varphi_m(x) \, dx = \sum_{l=1}^k \int_D c_l \varphi_l(x) \varphi_m(x) \, dx = c_m.$$

So the expression in (3.24) must be 0. We also showed that $Q(v_k,v_k) = \sum_{m=1}^k c_m^2 \lambda_m$.

Since $u \in \mathcal{F}_n$, $||\tilde{u}||_2 = 1$ and $c_1 = \cdots = c_{n-1} = 0$ so we obtain $\sum_{m=n}^{\infty} c_m^2 = 1$. Therefore for $k \geqslant n$ we get by (3.23)

$$Q(u,u) \geqslant Q(v_k, v_k) = \sum_{m=n}^{k} c_m^2 \lambda_m \geqslant \lambda_n \sum_{m=n}^{k} c_m^2.$$

Since $k \geqslant n$ is arbitrary, we conclude that $Q(u,u) \geqslant \lambda_n$. $\square$

With our variational formula established, our next goal is to prove an analogue of the Courant–Hilbert nodal domain theorem for the Cauchy process. We need the following definition. Each connected component of a set on which u_n has constant sign will be called *a nodal part for* u_n. The nodal domain for φ_n have already been defined in the introduction as a connected component of a set on which φ_n has a constant sign. It is important to keep in mind that a nodal part is a subset of H and that a nodal domain is a subset of D.

We will also need the following auxiliary fact. The proof of this fact is standard. We omit its proof.

Lemma 3.9. *Let $f \in L^1(\mathbb{R}^d)$ and assume that the Lebesgue measure in $\mathbb{R}^d$ of the set $\{x \in \mathbb{R}^d : f(x) \neq 0\}$ is positive. Let $u(x,t) = P_t f(x)$, $(x,t) \in H_+$. Then the Lebesgue measure in $\mathbb{R}^{d+1}$ of the set $\{(x,t) \in H_+ : u(x,t) = 0\}$ is zero.*

For any $A \subset H$ we will set $\tilde{A} = \{x \in D : (x,0) \in A\}$.

Lemma 3.10. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let A be a nodal part for u_n. Then $1_A u_n \in \mathcal{F}$ and*

$$\int_H |\nabla(1_A u_n)(x,t)|^2 \, dx \, dt = \lambda_n \int_{\tilde{A}} \varphi_n^2(x) \, dx.$$

Proof. If $(x,t) \in H_+$ and $\nabla(1_A u_n)(x,t)$ does not exists, then $(x,t) \in \partial A \cap H_+$ so $u_n(x,t) = (1_A u_n)(x,t) = 0$. Now the fact that $1_A u_n \in \mathcal{F}$ follows easily from the fact that $u_n \in \mathcal{F}$ and Lemma 3.9. Note also that

$$\int_H |\nabla(1_A u_n)(x,t)|^2 \, dx \, dt = \int_H \nabla(1_A u_n)(x,t) \nabla u_n(x,t) \, dx \, dt.$$

By Proposition 3.7 this equals

$$\begin{aligned} \lambda_n \int_D (1_A u_n)(x,0) \varphi_n(x) \, dx &= \lambda_n \int_D 1_A(x,0) \varphi_n^2(x) \, dx \\ &= \lambda_n \int_{\tilde{A}} \varphi_n^2(x) \, dx. \qquad \square \end{aligned}$$

The next result is an analogue of the Courant–Hilbert nodal domain theorem for our Steklov problem.

Theorem 3.11. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. The function u_n has no more than n nodal parts.*

Proof. Assume, for the purpose of contradiction, that u_n has at least $n+1$ nodal parts. We denote these sets by $A_1, \dots, A_{n+1}$. Observe that $1_{A_m} u_n \in \mathscr{F}$ and that $\widetilde{(1_{A_m} u_n)} = 1_{\tilde{A}_m} \tilde{u}_n$ for each $m = 1, \dots, n$. Put

$$u(x,t) = \sum_{m=1}^{n} b_m (1_{A_m} u_n)(x,t), \quad (x,t) \in H$$

so that

$$\tilde{u}(x) = \sum_{m=1}^{n} b_m (1_{\tilde{A}_m} \tilde{u}_n)(x), \quad x \in \mathbb{R}^d.$$

Let us choose $b_1, \dots, b_n \in \mathbb{R}$ such that $\tilde{u} \perp \varphi_1, \dots, \varphi_{n-1}$ and $||\tilde{u}||_2 = 1$. Such a choice is possible because $1_{\tilde{A}_1} \tilde{u}_n, \dots, 1_{\tilde{A}_n} \tilde{u}_n$ are linearly independent. From the linear independence it follows that

$$1 = ||\tilde{u}||_2^2 = \sum_{m=1}^{n} b_m^2 ||1_{\tilde{A}_m} \tilde{u}_n||_2^2.$$

By linearity and Lemma 3.10,

$$\begin{aligned} Q(u,u) &= \sum_{m=1}^{n} \sum_{k=1}^{n} Q(b_m 1_{A_m} u_n, b_k 1_{A_k} u_n) \\ &= \sum_{m=1}^{n} b_m^2 Q(1_{A_m} u_n, 1_{A_m} u_n) \\ &= \sum_{m=1}^{n} b_m^2 \lambda_n \int_{\tilde{A}_m} \varphi_n^2(x)\, dx \\ &= \lambda_n \sum_{m=1}^{n} b_m^2 ||1_{\tilde{A}_m} \tilde{u}_n||_2^2 \\ &= \lambda_n. \end{aligned} \tag{3.25}$$

For any $k \in \mathbb{N}$ and $(x,t) \in H$ put

$$v_k(x,t) = \sum_{m=1}^{k} c_m u_m(x,t),$$

where

$$c_m = \int_D \tilde{u}(x)\varphi_m(x)\,dx.$$

Then $v_k \in \mathscr{F}$ and so

$$Q(u,u) = Q(u - v_k, u - v_k) + Q(v_k, v_k) + 2Q(u - v_k, v_k). \tag{3.26}$$

The same argument as in the proof of Theorem 3.8 shows that $Q(u - v_k, v_k) = 0$ and $Q(v_k, v_k) = \sum_{m=1}^{k} c_m^2 \lambda_m$. Since $\tilde{u} \perp \varphi_1, \dots, \varphi_{n-1}$, $c_1 = \cdots = c_{n-1} = 0$. Since $||\tilde{u}||_2 = 1$, $\sum_{m=n}^{\infty} c_m^2 = 1$. Therefore for $k \geqslant n$ we obtain by (3.26)

$$Q(u,u) \geqslant Q(v_k, v_k) = \sum_{m=n}^{k} c_m^2 \lambda_m.$$

Let $N = \max\{m \in \mathbb{N} : \lambda_m = \lambda_n\}$. There are two cases to consider.

Case 1: There exists $m_0 > N$, $m_0 \in \mathbb{N}$ such that $c_{m_0}^2 > 0$. If this is so, then for any $k > m_0$ we obtain that

$$Q(u,u) \geqslant \sum_{m=n}^{k} c_m^2 \lambda_m > \lambda_n \sum_{m=n}^{k} c_m^2.$$

Since $\sum_{m=n}^{\infty} c_m^2 = 1$ and $k > m_0$ is arbitrary, it follows that $Q(u,u) > \lambda_n$ which is a contradiction to (3.25).

Case 2: For all $m > N$, $c_m = 0$. Then $\sum_{m=n}^{N} c_m^2 = 1$. In this case $Q(v_N, v_N) = \sum_{m=n}^{N} c_m^2 \lambda_m = \lambda_n$. On the other hand, we obtain

$$\begin{aligned} Q(u - v_N, u - v_N) &\geqslant \int_{A_{n+1}} |\nabla(u - v_N)(x,t)|^2\,dx\,dt. \\ &= \int_{A_{n+1}} |\nabla v_N(x,t)|^2\,dx\,dt, \end{aligned}$$

where we used the fact that $u(x,t) = 0$ for any $(x,t) \in A_{n+1}$. But $A_{n+1} \cap H_+$ is a non-empty open set and v_N is a non-trivial harmonic function on H_+. Thus

$$\int_{A_{n+1}} |\nabla v_N(x,t)|^2 > 0.$$

Hence by (3.26) we get

$$Q(u,u) = Q(u - v_N, u - v_N) + Q(v_N, v_N) > \lambda_n.$$

This again is a contradiction to (3.25) and completes the proof of the theorem. $\square$

With our variational characterization of λ_n and our Courant–Hilbert nodal domain theorem, we can now prove several estimates for λ_n which are similar to the classical estimates for the eigenvalues of the Laplacian. Such estimates will become useful in Section 5 below. To avoid confusion between eigenvalues of different domains, we will write $\lambda_n(D)$ in place of λ_n when the possibility of such confusion arises.

Proposition 3.12. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let A be a nodal part for u_n for the set D. Assume there exists a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ such that $\tilde{A} \subset \Omega$. Then*

$$\lambda_n(D) \geqslant \lambda_1(\Omega). \tag{3.27}$$

In particular, if $\tilde{A}$ is a bounded Lipschitz domain itself, this holds for $\Omega = \tilde{A}$.

Proof. Let $v(x,t) = ||1_{\tilde{A}}\tilde{u}_n||_2^{-1}(1_A u_n)(x,t)$, $(x,t) \in H$. Note that $v \in \mathscr{F}(\Omega)$. Moreover, $||\tilde{v}||_2 = ||1_{\tilde{A}}\tilde{u}_n||_2^{-1}||1_{\tilde{A}}\tilde{u}_n||_2 = 1$. Hence $v \in \mathscr{F}_1(\Omega)$. By Theorem 3.8,

$$\lambda_1(\Omega) = \inf_{u \in \mathscr{F}_1(\Omega)} Q(u,u) \leqslant Q(v,v).$$

On the other hand, by Lemma 3.10,

$$\begin{aligned} Q(v,v) &= ||1_{\tilde{A}}\tilde{u}_n||_2^{-2} \int_H |\nabla(1_A u_n)(x,t)|^2 \, dx \, dt \\ &= ||1_{\tilde{A}}\tilde{u}_n||_2^{-2} \lambda_n(D) \int_{\tilde{A}} \varphi_n^2 \, dx = \lambda_n(D). \end{aligned}$$

This proves the proposition. $\square$

The following result is an analog of the Reylich-Ritz mini-max formula.

Proposition 3.13. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let L be a non-empty linear subspace of $\mathscr{F}$ and $\tilde{L} = \{\tilde{u} : u \in L\}$. Put*

$$R(L) = \sup\left\{ \int_H |\nabla u(x,t)|^2 \, dx \, dt : u \in L, \ ||\tilde{u}||_2 = 1 \right\}. \tag{3.28}$$

Then

$$\lambda_n = \inf\{R(L) : \dim(\tilde{L}) = n\}, \quad n \in \mathbb{N}.$$

Proof. Let $L_n = \mathrm{Span}\{u_1, \ldots, u_n\}$. Then $\tilde{L}_n = \mathrm{Span}\{\varphi_1, \ldots, \varphi_n\}$ and $\dim(\tilde{L}_n) = n$. Hence

$$\inf\{R(L) : \dim(\tilde{L}) = n\} \leqslant R(L_n) = \lambda_n. \tag{3.29}$$

On the other hand, let L be an arbitrary linear subspace of $\mathscr{F}$ such that $\dim(\tilde{L}) = n$. Then there exist $v_1, \dots, v_n \in L$ such that

$$\tilde{L} = \mathrm{Span}\{\tilde{v}_1, \dots, \tilde{v}_n\}.$$

Put $w = c_1 v_1 + \cdots + c_n v_n$. We have $\tilde{w} = c_1 \tilde{v}_1 + \cdots + c_n \tilde{v}_n$. Let us choose $c_1, \dots, c_n$ so that $\tilde{w} \perp \varphi_1, \dots, \varphi_{n-1}$ and $\|\tilde{w}\|_2 = 1$. Such a choice is possible because of the linear independence of $\tilde{v}_1, \dots, \tilde{v}_n$. Then $w \in \mathscr{F}_n$ and it follows that

$$R(L) \geqslant Q(w,w) \geqslant \inf_{u \in \mathscr{F}_n} Q(u,u) = \lambda_n. \tag{3.30}$$

The proposition follows from (3.29) and (3.30). $\square$

Let $D \subset \mathbb{R}^d$ be a bounded connected Lipschitz domain. Recall that $\{\psi_n, \mu_n\}$ is the solution of the Dirichlet eigenvalue problem

$$\begin{cases} \Delta \psi_n(x) = -\mu_n \psi_n(x), & x \in D, \\ \psi_n(x) = 0, & x \in \partial D \end{cases} \tag{3.31}$$

as discussed in the Introduction. We assume, as we did in the Introduction, that $\{\psi_n\}_{n=1}^{\infty}$ is an orthonormal basis in $L^2(D)$ and recall that $0 < \mu_1 < \mu_2 \leqslant \mu_3 \leqslant \cdots$ and $\mu_n \to \infty$ as $n \to \infty$. Multiplying the first part of the equation in (3.31) by ψ_m and integrating gives

$$\int_D \psi_m(x) \Delta \psi_n(x)\, dx = -\mu_n \int_D \psi_m(x) \psi_n(x)\, dx.$$

Integrating by parts (apply Green's theorem) and using the orthonormal properties of the functions ψ_n, it follows that

$$\int_D \nabla_x \psi_n(x) \nabla_x \psi_m(x)\, dx = \begin{cases} \mu_n & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases} \tag{3.32}$$

where

$$\nabla_x = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d} \right).$$

This identity in fact holds for a wider class of domains other than Lipschitz but for our purpose this will suffice. Using this we have the following application of Proposition 3.13 which gives a comparison of the higher eigenvalues of the Dirichlet Laplacian to those of the Cauchy process. A comparison between the first eigenvalue of any symmetric stable process of order $0 < \alpha < 2$ and the first eigenvalue for the Laplacian is given in [2]. Our result here shows that comparison remains valid for the full spectrum in the case of the Cauchy process. We of course expect this to be the case for all $\alpha \in (0,2)$ as well, with upper bound $\mu_n^{\alpha/2}$ as in [2].

Theorem 3.14. *Let $D \subset \mathbb{R}^d$ be a bounded connected Lipschitz domain. Then for any $n \in \mathbb{N}$*

$$\lambda_n \leqslant \sqrt{\mu_n}.$$

Proof. We extend $\psi_n(x)$ to all of $\mathbb{R}^d$ by setting $\psi_n(x) \equiv 0$ for $x \in D^c$. Let $v_n(x,t) = \exp(-\sqrt{\mu_n}t)\psi_n(x)$, $(x,t) \in H$, $n \in \mathbb{N}$. Then

$$\begin{aligned}|\nabla v_n(x,t)|^2 &= |\nabla_x v_n(x,t)|^2 + \left|\frac{\partial v_n}{\partial t}(x,t)\right|^2 \\ &= \exp(-2\sqrt{\mu_n}t)|\nabla_x \psi_n(x)|^2 + \mu_n \exp\big(-2\sqrt{\mu_n}t\big)(\psi_n(x))^2.\end{aligned}$$

Integrating gives

$$\begin{aligned}\int_H |\nabla v_n(x,t)|^2\,dx\,dt &= \int_{D\times[0,\infty)} |\nabla v_n(x,t)|^2\,dx\,dt \\ &= \int_0^\infty \exp\big(-2\sqrt{\mu_n}t\big)\,dt \int_D |\nabla_x \psi_n(x)|^2\,dx \\ &\quad + \mu_n \int_0^\infty \exp\big(-2\sqrt{\mu_n}t\big)\,dt \int_D (\psi_n(x))^2\,dx = \sqrt{\mu_n}.\end{aligned}$$

For $m \neq n$, $m,n \in \mathbb{N}$,

$$\begin{aligned}\int_H \nabla v_n(x,t)\nabla v_m(x,t)\,dx\,dt = &\int_{D\times[0,\infty)} \nabla_x v_n(x,t)\nabla_x v_m(x,t)\,dx\,dt \\ &+ \int_{D\times[0,\infty)} \frac{\partial v_n}{\partial t}(x,t)\frac{\partial v_m}{\partial t}(x,t)\,dx\,dt.\end{aligned}$$

The first integral on the right-hand side equals

$$\int_0^\infty \exp\big(-t\big(\sqrt{\mu_n}+\sqrt{\mu_m}\big)\big)\,dt \int_D \nabla_x \psi_n(x)\nabla_x \psi_m(x)\,dx$$

and the second integral equals

$$\sqrt{\mu_n}\sqrt{\mu_m}\int_0^\infty \exp(-t(\sqrt{\mu_n}+\sqrt{\mu_m}))\,dt \int_D \psi_n(x)\psi_m(x)\,dx.$$

By the orthogonality of the functions ψ_n and (3.23), both of these quantities are 0. Note that $v_n \in \mathscr{F}$, $n \in \mathbb{N}$. Let $L_n' = \{v_1, \dots, v_n\}$. Then $\tilde{L}_n' = \{\psi_1, \dots, \psi_n\}$ and $\dim(\tilde{L}_n') = n$. Let $w = c_1 v_1 + \dots + c_n v_n$ be an arbitrary function from L_n' such that $||\tilde{w}||_2 = 1$. Then

$$1 = ||\tilde{w}||_2^2 = \sum_{m=1}^n c_m^2 ||\psi_m||_2^2 = \sum_{m=1}^n c_m^2.$$

Therefore

$$Q(w,w) = \sum_{m=1}^{n} \sum_{l=1}^{n} c_m c_l Q(v_m, v_l) = \sum_{m=1}^{n} c_m^2 \sqrt{\mu_m} \leqslant \sqrt{\mu_n} \sum_{m=1}^{n} c_m^2 = \sqrt{\mu_n}.$$

Hence $R(L_n') = \sqrt{\mu_n}$. By Proposition 3.13 we obtain

$$\lambda_n = \inf\{R(L) : \dim(\tilde{L}) = n\} \leqslant R(L_n') = \sqrt{\mu_n},$$

which completes the proof. $\square$

We now derive several results which will be needed in Section 5 when we study the shape of φ_2 and the zeros of φ_n for the interval $(-1, 1)$. The next results do not use the variational characterization of λ_n but follow instead more directly from the fact that u_n is a solution of the mixed Steklov eigenvalue problem (1.7)–(1.9) (Theorem 1.1). We will use the notation of Proposition 3.1.

Proposition 3.15. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Fix $n \in \mathbb{N}$. Assume that $r_n(x) \geqslant 0$ for all $x \in \mathbb{R}^d$. Then*

(i) *The Lebesgue measure of the set $\{x \in \mathbb{R}^d : r_n(x) > 0\}$ is positive. In particular, for all $(x,t) \in H_+$ we have $P_t r_n(x) > 0$.*
(ii) *Suppose there is $x_0 \in \mathbb{R}^d$ and $t_0 \geqslant 0$ such that $u_n(x_0, t_0) \geqslant 0$. Then for all $t > t_0$, $u_n(x_0, t) > 0$.*

Proof. (i) Suppose on the contrary that the Lebesgue measure of the set $\{x \in \mathbb{R}^d : r_n(x) > 0\}$ is zero. Then for all $(x,t) \in H_+$, $P_t r_n(x) = 0$. Hence by Proposition 3.2(iii) we obtain

$$\frac{\partial u_n}{\partial t}(x,t) = -\lambda_n u_n(x,t), \quad (x,t) \in H_+.$$

Note that $u_n(x,0) = 0$ for $x \in D^c$. So, from the above it follows that $u_n(x,t) = 0$ for all $x \in D^c$ and $t > 0$. The function $u_n = P_t \varphi_n$ is harmonic in H_+. Since it vanishes in $D^c \times [0, \infty)$, it must vanish in H_+. This gives a contradiction to the fact that φ_n is not trivial.

(ii) This will follow, as we shall see, from the weaker statement:

(ii′) Suppose there is $x_0 \in \mathbb{R}^d$ and $t_0 \geqslant 0$ such that $u_n(x_0, t_0) > 0$. Then $u_n(x_0, t) > 0$ for all $t \geqslant t_0$.

To see this, suppose there exists $t > t_0$ such that $u_n(x_0, t) = 0$. Let $t_1 = \inf\{t > t_0 : u_n(x_0, t) = 0\}$. It follows that $t_0 < t_1 < \infty$, $u_n(x_0, t) > 0$ for $t \in [t_0, t_1)$ and $u_n(x_0, t_1) = 0$. Then

$$\frac{\partial u_n}{\partial t}(x_0, t_1) = \lim_{h \to 0^+} \frac{u_n(x_0, t_1 - h) - u_n(x_0, t_1)}{-h} \leqslant 0.$$

On the other hand, $u_n(x_0, t_1) = 0$ and $P_{t_1} r_n(x_0) > 0$ so by Proposition 3.2(iii) we obtain

$$\frac{\partial u_n}{\partial t}(x_0, t_1) = -\lambda_n u_n(x_0, t_1) + P_{t_1} r_n(x_0) > 0,$$

which gives a contradiction. This proves (ii$'$).

With the weaker form (ii$'$) proved, we have to consider the case $u_n(x_0, t_0) = 0$. Let $t_1 = \inf\{t > t_0 : u_n(x_0, t) > 0\}$ where we put $t_1 = \infty$ if the set $\{t > t_0 : u_n(x_0, t) > 0\}$ is empty. If $t_1 = t_0$, that is, if there exists a sequence $\{s_k\}_{k=1}^{\infty}$ such that $s_k > t_0$ and $\lim_{k\to\infty} s_k = t_0$, $u_n(x_0, s_k) > 0$, then by (ii$'$) we obtain that for all $t > t_0$, $u_n(x_0, t) > 0$. So, we can assume that $t_1 > t_0$. Let us take t_2 such that $t_0 < t_2 < t_1$. For all $t \in [t_0, t_2]$ we have $u_n(x_0, t) \leqslant 0$. By the mean value theorem there exists $\xi \in (t_0, t_2)$ such that

$$u_n(x_0, t_2) - u_n(x_0, t_0) = (t_2 - t_0)\frac{\partial u_n}{\partial t}(x_0, \xi).$$

Since $u_n(x_0, t_2) \leqslant 0$ and $u_n(x_0, t_0) = 0$, $\frac{\partial u_n}{\partial t}(x_0, \xi) \leqslant 0$. We also have $u_n(x_0, \xi) \leqslant 0$, $(\xi \in (t_0, t_2))$. So, by Proposition 3.2(iii) we get

$$\frac{\partial u_n}{\partial t}(x_0, \xi) = -\lambda_n u_n(x_0, \xi) + P_{\xi} r_n(x_0) > 0,$$

which gives a contradiction. □

We need the following well-known lemma which can be proved from the reflection principle and the Harnack inequality, or also from the general boundary Harnack Principle in Lipschitz domains.

Lemma 3.16. *Let $r, h > 0$ and let $\mathscr{C} = \{(x, t) \in H : |x|^2 \leqslant r,\ t \in [0, h]\}$ be a cylinder in H. Let $u : \mathscr{C} \to \mathbb{R}$ be non-negative, continuous in $\mathscr{C}$ and harmonic in the interior of $\mathscr{C}$. Assume in addition that u is positive on $\{(x, t) \in H : |x|^2 \leqslant r,\ t = h\}$. Then there exists $c = c(r, h, u) > 0$ such that*

$$u(0, t) \geqslant c\, t, \quad t \in [0, h].$$

Proposition 3.17. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $x_0 \in D$ and $r, h > 0$. Fix $n \in \mathbb{N}$. Assume that u_n is positive on the set $\{(x, t) \in H : |x - x_0|^2 \leqslant r^2,\ t \in (0, h]\}$. Then $\varphi_n(x_0) = u_n(x_0, 0) > 0$.*

Proof. On the contrary, assume that $u_n(x_0, 0) = 0$. By Lemma 3.16, we obtain $u_n(x_0, t) \geqslant c\, t$ for all $t \in [0, h]$, where $c > 0$ depends on u_n, r, h, x_0. Hence

$$\frac{\partial u_n}{\partial t}(x_0, 0) = \lim_{t\to 0^+} \frac{u_n(x_0, t) - u_n(x_0, 0)}{t} \geqslant \lim_{t\to 0^+} \frac{ct}{t} = c > 0.$$

On the other hand,

$$\frac{\partial u_n}{\partial t}(x_0, 0) = -\lambda_n u_n(x_0, 0) = 0,$$

which gives a contradiction. $\square$

In what follows, we will often refer to "smooth bounded connected domains." By this we will mean a bounded connected domain which is at least C^2. Often these results hold for more general domains but their proofs are more technical. For our purposes, smooth domains suffice. The next theorem is an auxiliary result but it will be crucial in identifying the second eigenfunction for $D = (-1, 1)$ in Section 5.

Theorem 3.18. *Let D be a bounded Lipschitz domain. Let A be a nodal part for u_n for the set D. Assume there exists a smooth bounded connected domain Ω with $\Omega \subset D$ such that $A \subset \Omega \times [0, \infty)$. Then we have*

$$\lambda_n(D) \geqslant \sqrt{\mu_1(\Omega)},$$

where $\mu_1(\Omega)$ is the solution of the Dirichlet eigenvalue problem (3.31) *for the domain Ω.*

In order to prove this theorem we will have to obtain a result similar to Theorem 3.8 but for $\Omega \times [0, \infty)$ instead of H. Namely, we will show that for each $n \in \mathbb{N}$,

$$\sqrt{\mu_n} = \inf_{u \in \mathcal{G}_n} \int_0^\infty \int_\Omega |\nabla u(x, t)|^2 \, dx \, dt,$$

for a suitably chosen class of functions $\mathcal{G}_n$, where μ_n are eigenvalues for the Dirichlet problem (3.31) for Ω. We first introduce the class of functions $\mathcal{G}(\Omega) = \mathcal{G}$ for the variational characterization of $\sqrt{\mu_n}$.

Definition 3.2. Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded connected domain. We define $\mathcal{G}(\Omega)$ to be the collection of all finite linear combinations of functions $u : \Omega \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(i) u is continuous and bounded on $\Omega \times [0, \infty)$ and for all $t > 0$ there exists a constant c_t such that for all $x \in \Omega$,

$$|u(x, t)| \leqslant c_t \, \delta_\Omega(x).$$

(ii) $\nabla u(x, t)$ exists for almost all $(x, t) \in \Omega \times (0, \infty)$ and ∇u is a measurable function. If $(x, t) \in \Omega \times (0, \infty)$ and $\nabla u(x, t)$ does not exists, then $u(x, t) = 0$. Moreover for all $t > 0$, there is a c'_t such that for all $x \in \Omega$,

$$|\nabla u(x, t)| \leqslant c'_t,$$

for any $(x,t)\in\Omega\times(0,\infty)$ for which $\nabla u(x,t)$ exists.

(iii)
$$\int_\Omega u^2(x,0)\,dx<\infty$$

and

(iv)
$$\int_0^\infty\int_\Omega|\nabla u(x,t)|^2\,dx\,dt<\infty.$$

We will just write $\mathcal{G}$ for $\mathcal{G}(\Omega)$ whenever Ω is fixed. Recall that $\nabla=\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_d},\frac{\partial}{\partial t}\right)$ and that $\nabla_x=\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_d}\right)$. As before, set $v_n(x,t)=\exp(-\sqrt{\mu_n}t)\psi_n(x)$ for $(x,t)\in\Omega\times[0,\infty)$ where ψ_n is the Dirichlet eigenfunction corresponding to μ_n.

Lemma 3.19. *Let $\Omega\subset\mathbb{R}^d$ be a smooth bounded connected domain. Then*

(i) *For any $n\in\mathbb{N}$, there exists constant $C_1(n,\Omega)$ such that for all $(x,t)\in\Omega\times[0,\infty)$,*
$$|\nabla v_n(x,t)|\leqslant C_1(n,\Omega).$$

(ii) *If $u\in\mathcal{G}$, $t>0$, and $n\in\mathbb{N}$, then there is a constant $C_2(u,t,n,\Omega)$ such that for $x\in\Omega$ and $i=1,2,\dots,d$,*
$$\left|u(x,t)\frac{\partial^2 v_n}{\partial x_i^2}(x,t)\right|\leqslant C_2(u,t,n,\Omega).$$

(iii) *For any $n\in\mathbb{N}$ we have $v_n\in\mathcal{G}$ and*
$$\int_0^\infty\int_\Omega|\nabla v_n(x,t)|^2\,dx\,dt=\sqrt{\mu_n}. \tag{3.33}$$

Proof. As before, a direct calculation gives,
$$|\nabla v_n(x,t)|\leqslant\exp(-\sqrt{\mu_n}t)|\nabla_x\psi_n(x)|+\sqrt{\mu_n}\exp(-\sqrt{\mu_n}t)|\psi_n(x)|.$$

By the smoothness of $\partial\Omega$ (C^2 is enough here) and intrinsic ultracontractivity we have
$$|\psi_n(x)|\leqslant c(n,\Omega)\psi_1(x)\leqslant c(n,\Omega)\delta_\Omega(x),\quad x\in\Omega. \tag{3.34}$$

Here we recall our convention that constants may change their value from one use to the next even on the same line. By [25], Theorem 1
$$|\nabla\psi_n(x)|\leqslant c(n,\Omega)\,|\psi_n(x)|\,\delta_\Omega^{-1}(x)\leqslant c(n,\Omega),\quad x\in\Omega,$$

and this proves (i).

One more application of the upper bound of [25] gives

$$\left|\frac{\partial^2\psi_n}{\partial x_i^2}(x)\right| \leqslant c(n,\Omega)\left|\frac{\partial\psi_n}{\partial x_i}(x)\right|\frac{1}{\delta_\Omega(x)} \leqslant c(n,\Omega)\frac{1}{\delta_\Omega(x)}$$

and (ii) follows by (i) in Definition 3.2. We point out that Theorem 1, [25] is stated for $d \geqslant 3$. For $d=1$ the only smooth bounded connected domain Ω is an interval so the above inequalities follow from explicit expressions for ψ_n. For $d=2$ we may use Theorem 1, [25] by adding extra dimension. In fact for $d=2$ Theorem 1, [25] may be used for $D\times(0,R)$, for some $R>0$ and function $\hat{\psi}_n(x_1,x_2,x_3)=\psi_n(x_1,x_2)$, $(x_1,x_2)\in D$, $x_3\in(0,R)$.

Identity (3.33) was proved in the proof of Theorem 3.14. Continuity and boundedness of v_n are clear. The second part of condition (i) in Definition 3.2 follows from (3.34). Condition (ii) is satisfied by (i) in this lemma. Conditions (iii) and (iv) of Definition 3.2 follow easily. □

Next, we prove a result similar to Proposition 3.7.

Proposition 3.20. *Let $\Omega\subset\mathbb{R}^d$ be a smooth bounded connected domain and $u\in\mathcal{G}$. Then for any $n\in\mathbb{N}$,*

$$\int_0^\infty\int_\Omega \nabla u(x,t)\nabla v_n(x,t)\,dx\,dt = \sqrt{\mu_n}\int_\Omega u(x,0)\psi_n(x)\,dx. \tag{3.35}$$

We interpret this identity as the statement that Green's theorem can be applied in the sense that

$$\begin{aligned}\int_0^\infty\int_\Omega \nabla u(x,t)\nabla v_n(x,t)\,dx\,dt = &-\int_0^\infty\int_\Omega u(x,t)\Delta v_n(x,t)\,dx\,dt\\ &-\int_0^\infty\int_{\partial\Omega} u(x,t)\frac{\partial v_n}{\partial \nu}(x,0)\,dx\,dt\\ &-\int_\Omega u(x,0)\frac{\partial v_n}{\partial t}(x,0)\,dx.\end{aligned} \tag{3.36}$$

Here, $\frac{\partial}{\partial \nu}$ is the inward normal derivative at $\partial\Omega$. Note also (see (3.39)) that $\Delta v_n(x,t)=0$ for $(x,t)\in\Omega\times(0,\infty)$. We will show (3.35) and not (3.36). Eq. (3.36) is only an interpretation and is not fully precise. For example, the functions u and v_n are not defined on $\partial\Omega$ so the integral over $\partial\Omega$ should be understood in the sense of limits.

Proof. The integral on the left-hand side of (3.35) equals

$$\int_0^\infty\int_\Omega \nabla_x u(x,t)\nabla_x v_n(x,t)\,dx\,dt + \int_0^\infty\int_\Omega \frac{\partial u}{\partial t}(x,t)\frac{\partial v_n}{\partial t}(x,t)\,dx\,dt = \mathrm{I}+\mathrm{II}.$$

Since u, v_n satisfies (iv) in Definition (3.2), it follows that

$$\int_0^\infty \left| \frac{\partial u}{\partial t}(x,t) \frac{\partial v_n}{\partial t}(x,t) \right| dt < \infty$$

for almost all $x \in \Omega$. For such an x,

$$\int_0^\infty \frac{\partial u}{\partial t}(x,t) \frac{\partial v_n}{\partial t}(x,t)\, dt$$
$$= \left[u(x,t) \frac{\partial v_n}{\partial t}(x,t) \right]_{t=0}^{t=\infty} - \int_0^\infty u(x,t) \frac{\partial^2 v_n}{\partial t^2}(x,t)\, dt.$$

It follows that

$$\mathrm{II} = \sqrt{\mu_n} \int_\Omega u(x,0)\psi_n(x)\, dx - \int_0^\infty \int_\Omega u(x,t) \frac{\partial^2 v_n}{\partial t^2}(x,t)\, dx\, dt = \mathrm{III} - \mathrm{IV}.$$

Hence, to prove the proposition it remains to show that $\mathrm{I} - \mathrm{IV} = 0$. In fact, it is enough to show that for each $t > 0$,

$$\mathrm{V} - \mathrm{VI} = \int_\Omega \nabla_x u(x,t) \nabla_x v_n(x,t)\, dx - \int_\Omega u(x,t) \frac{\partial^2 v_n}{\partial t^2}(x,t)\, dx = 0. \tag{3.37}$$

Notice that by condition (ii) in Definition 3.2, for each $t > 0$ the integral V is absolutely convergent. Similarly, by the boundedness of u (condition (i) in Definition 3.2) and the explicit expression for v_n, integral VI is absolutely convergent.

Fix $t > 0$. Extend u and v_n to all of H by putting $u(x,t) = 0$, $v_n(x,t) = 0$ for $(x,t) \notin \Omega \times [0,\infty)$. We have

$$\mathrm{V} = \sum_{i=1}^d \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \frac{\partial u}{\partial x_i}(x,t) \frac{\partial v_n}{\partial x_i}(x,t)\, dx_1 \ldots dx_d.$$

Now we need to show that

$$\int_{-\infty}^\infty \frac{\partial u}{\partial x_i}(x,t) \frac{\partial v_n}{\partial x_i}(x,t)\, dx_i = -\int_{-\infty}^\infty u(x,t) \frac{\partial v_n}{\partial x_i^2}(x,t)\, dx_i. \tag{3.38}$$

Observe that the integral on the left is absolutely convergent by condition (ii) in Definition 3.2 and the integral on the right is well defined by Lemma 3.19(ii). The justification of (3.38) is almost the same as the justification of (3.10) in the proof of Proposition 3.4 and therefore we omit it. It follows that

$$\mathrm{V} = -\sum_{i=1}^d \int_\Omega u(x,t) \frac{\partial v_n}{\partial x_i^2}(x,t)\, dx.$$

On the other hand,

$$\Delta v_n(x,t) = \sum_{i=1}^{d} \frac{\partial v_n}{\partial x_i^2}(x,t) + \frac{\partial v_n}{\partial t^2}(x,t) = 0. \tag{3.39}$$

This gives (3.37) and proves the proposition. $\square$

We now define the "variational spaces" for the set $\Omega \times [0,\infty)$, where Ω is a smooth bounded connected domain. For any $u : \Omega \times [0,\infty) \to \mathbb{R}$, we put $\tilde{u}(x) = u(x,0)$, $x \in \Omega$ and $||\tilde{u}||_\Omega = (\int_\Omega \tilde{u}^2(x)\,dx)^{1/2}$. Let $\mathscr{G}_1(\Omega) = \{u \in \mathscr{G}(\Omega) : ||\tilde{u}||_\Omega = 1\}$ and for $n \geqslant 2$, let

$$\mathscr{G}_n(\Omega) = \{u \in \mathscr{G}(\Omega) : \tilde{u} \perp \psi_1, \dots \psi_{n-1};\ ||\tilde{u}||_\Omega = 1\}$$

and as before, we will write $\mathscr{G}_n(\Omega)$ for $\mathscr{G}_n$ when the set Ω is well understood.

Proposition 3.21. *Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded connected domain. Then*

$$\sqrt{\mu_n} = \inf_{u \in \mathscr{G}_n} \int_0^\infty \int_\Omega |\nabla u(x,t)|^2\,dx\,dt,$$

for all $n \in \mathbb{N}$.

This proposition follows from Proposition 3.20 exactly in the same way as Theorem 3.8 follows from Proposition 3.7 (see the proof of Theorem 3.8) and we leave the details to the reader.

Proof of Theorem 3.18. For $(x,t) \in \Omega \times [0,\infty)$, consider the function

$$u(x,t) = \left(\int_{\tilde{A}} \varphi_n^2(x)\,dx\right)^{-1/2} (1_A u_n)(x,t).$$

By Lemma 3.10,

$$\begin{aligned}\int_0^\infty \int_\Omega |\nabla u(x,t)|^2\,dx\,dt &= \int_A |\nabla u(x,t)|^2\,dx\,dt \\ &= \left(\int_{\tilde{A}} \varphi_n^2(x)\,dx\right)^{-1} \int_A |\nabla(1_A u_n)(x,t)|^2\,dx\,dt = \lambda_n(D).\end{aligned}$$

Since $A \subset \Omega \times [0,\infty)$, we see that $u \in \mathscr{G}$. Note that $||\tilde{u}||_\Omega = 1$, so $u \in \mathscr{G}_1$. Hence by Proposition 3.21 we obtain

$$\sqrt{\mu_1(\Omega)} \leqslant \int_0^\infty \int_\Omega |\nabla u(x,t)|^2\,dx\,dt = \lambda_n(D),$$

proving the theorem. $\square$

4. Eigenfunctions and eigenvalues

In this section we will derive several results which will be of use in Section 5 and which are also of independent interest. Our first result, the real analyticity of eigenfunctions, is a basic regularity results that we believe should be known, and as pointed out to us by A. Sá Barreto, it may follow from general considerations of pseudo-differential operators as in [17]. However, we have not been able to find an appropriate reference in the literature for it. Therefore we provide the simple, although technical, proof here. We point out that it is possible to generalize this result to all $\alpha \in (0, 2)$ but such a proof would demand more technical details. For simplicity, and because our main application here is to the Cauchy process we restrict ourselves to $\alpha = 1$. In a similar fashion, our second result (Theorem 4.3) which gives the existence of an antisymmetric eigenfunction φ_*, could be generalized to $\alpha \in (0, 2)$ and the assumptions on the domain in Theorem 4.3 such as Lipschitz boundary and connectedness of the domain are not necessary. Such assumptions make the arguments less technical and give the results we will need in our applications.

Theorem 4.1. *Let $D \subset \mathbb{R}^d$ be a bounded domain. The Cauchy eigenfunctions φ_n are real analytic in D.*

We need some auxiliary facts and additional notation. Let Λ be the set of all multi-index $\beta = (\beta_1, \dots, \beta_d)$ with $\beta_i \in \{0, 1, 2, \dots\}$ and as usual set $||\beta|| = \beta_1 + \cdots + \beta_d$. For any $f : \mathbb{R}^d \to \mathbb{R}$ and $\beta \in \Lambda$ set

$$D_x^{\beta} f(x) = D_x^{\beta_1, \dots, \beta_d} f(x) = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_d}}{\partial x_1^{\beta_d}} f(x)$$

whenever all the derivatives exist.

We will need some estimates on $D_x^{\beta} p(s, x, y)$. Set $F(x) = (s^2 + |x|^2)^{-(d+1)/2}$, $x \in \mathbb{R}^d$, $s > 0$.

Lemma 4.2. *For any $\beta \in \Lambda$,*

$$D_x^{\beta} F(x) = w_{\beta}(x)(s^2 + |x|^2)^{-||\beta|| - (d+1)/2}, \tag{4.1}$$

where $w_{\beta}(x) = \sum_{\gamma \in \Lambda : ||\gamma|| \leqslant ||\beta||} c_{\gamma,\beta} x_1^{\gamma_1} \dots x_d^{\gamma_d}$ and $c_{\gamma,\beta} \in \mathbb{R}$. For $n = 0, 1, 2, \dots$ set

$$a_n = \max_{\beta \in \Lambda : ||\beta|| = n} \left(\sum_{\gamma \in \Lambda : ||\gamma|| \leqslant ||\beta||} |c_{\gamma,\beta}| \right).$$

Then for any $s \leqslant 1$ and $n = 0, 1, 2 \dots$

$$a_n \leqslant (d + 3)^n (n!). \tag{4.2}$$

Also, for any $\beta \in \Lambda$, $s \leqslant 1$ and $x \in \mathbb{R}^d$,

$$|D_x^{\beta} F(x)| \leqslant \max(1, |x|^{-2\|\beta\|-d-1})(d+3)^{\|\beta\|}(\|\beta\|!). \tag{4.3}$$

In particular for $\beta \in \Lambda$, $s \leqslant 1$, $x, y \in \mathbb{R}^d$ we obtain

$$|D_x^{\beta} p(s,x,y)| \leqslant c_d \max(1, |x-y|^{-2\|\beta\|-d-1})(d+3)^{\|\beta\|}(\|\beta\|!). \tag{4.4}$$

Proof. We will prove (4.1) and (4.2) by induction. The proof is completely elementary based on our explicit expression for F. We present it here for completeness. Of course, for $\beta = (0, \dots, 0)$ both formulas are true. Assume that (4.1) and (4.2) are true for some $\beta \in \Lambda$. For any $i \in \{1, \dots, d\}$ we see that $\frac{\partial (D_x^{\beta} F)}{\partial x_i}(x)$ is equal to

$$\begin{aligned}&\left[(-\|\beta\| - (d+1)/2)2x_i w_{\beta}(x) + \frac{\partial w_{\beta}}{\partial x_i}(x)(s^2 + x_1^2 + \cdots + x_d^2)\right]\\ &\quad \times (s^2 + |x|^2)^{-\|\beta\|-1-(d+1)/2}.\end{aligned} \tag{4.5}$$

This justifies the induction step for (4.1). Eq. (4.5) and the assumption that $s \leqslant 1$ also gives

$$\begin{aligned}a_{n+1} &\leqslant (2\|\beta\| + (d+1))a_n + \|\beta\|(d+1)a_n\\ &= a_n(\|\beta\|(d+3) + d + 1) \leqslant a_n(\|\beta\| + 1)(d+3),\end{aligned}$$

and (4.2) follows.

Now (4.3) will follow from (4.1) and (4.5). In fact, for $s \leqslant 1$ we have

$$|w_{\beta}(x)| \leqslant \sum_{\gamma \in \Lambda\,:\,\|\gamma\| \leqslant \|\beta\|} |c_{\gamma,\beta}||x_1|^{\gamma_1} \dots |x_d|^{\gamma_d} \tag{4.6}$$

and

$$(s^2 + |x|^2)^{-\|\beta\|-(d+1)/2} \leqslant |x|^{-2\|\beta\|-d-1}. \tag{4.7}$$

Also, if $|x| \leqslant 1$ then the right-hand side of (4.6) is bounded above by

$$\sum_{\gamma \in \Lambda\,:\,\|\gamma\| \leqslant \|\beta\|} |c_{\gamma,\beta}| \leqslant a_{\|\beta\|}$$

and (4.3) follows. On the other hand, if $|x| > 1$ then the right-hand side of (4.6) is no larger than $|x|^{\|\beta\|} a_{\|\beta\|}$. This also gives (4.3). Eq. (4.4) follows trivially from (4.3) and our formula for $p(s,x,y)$ as given in (1.1). $\square$

Proof of Theorem 4.1. Fix $t>0$ and $k\in\mathbb{N}$. For any $x\in D$, $t>0$ and $k\in\mathbb{N}$,

$$e^{-\lambda_k t}\varphi_k(x) = P_t^D\varphi_k(x) = P_t\varphi_k(x) - \int_D r_D(t,x,y)\varphi_k(y)\,dy,$$

where $r_D(t,x,y)$ is given by (2.1). Of course, the function $x\to P_t\varphi_k(x)$, is real analytic in D. We must prove that the function

$$S_D\varphi_k(x) = \int_D r_D(t,x,y)\varphi_k(y)\,dy$$

is also real analytic in D. Fix $0<t\leqslant 1$. By (2.1) and the fact that $r_D(t,x,y)=r_D(t,y,x)$,

$$S_D\varphi_k(x) = \int_D E^y[t>\tau_D; p(t-\tau_D,x,X(\tau_D))]\varphi_k(y)\,dy, \quad x\in D.$$

Fix an arbitrary $z\in D$ and $r\in(0,\delta_D(z)/2]$. We claim that

$$D_x^\beta(S_D\varphi_k)(x) = \int_D E^y[t>\tau_D; D_x^\beta p(t-\tau_D,x,X(\tau_D))]\varphi_k(y)\,dy, \tag{4.8}$$

for any $\beta\in\Lambda$ and $x\in B(z,r)$. In particular we claim that the left-hand side of (4.8) is well defined. On the set $\{t>\tau_D\}$, we get by (4.4) (recall $t\in(0,1]$) that $|D_x^\beta p(t-\tau_D,x,X(\tau_D))|$ is bounded above by

$$c_d\max(1,|x-X(\tau_D)|^{-2||\beta||-d-1})(d+3)^{||\beta||}(||\beta||!).$$

But $|x-X(\tau_D)|\geqslant\delta_D(z)/2$ for $x\in B(z,r)$, $r\in(0,\delta_D(z)/2]$. Hence $D_x^\beta p(t-\tau_D,x,X(\tau_D))$ is bounded on $B(z,r)$. Recall also that φ_k is bounded on D. This gives that $D_x^\beta(S_D\varphi_k)(x)$ is well defined for $x\in B(z,r)$ and that (4.8) holds. Moreover, for $x\in B(z,r)$ we get

$$|D_x^\beta(S_D\varphi_k)(x)|\leqslant c(D,k)\max(1,(\delta_D(z)/2)^{-2||\beta||-d-1}) \times (d+3)^{||\beta||}(||\beta||!). \tag{4.9}$$

Thus the function $(S_D\varphi_k)$ is C^∞ in $B(z,r)$. We may therefore expand this function into its Taylor's series on $B(z,r)$ about the point z and we must show that the remainder goes to zero uniformly in $B(z,r)$. Let us denote this remainder by $R_n(S_D\varphi_k)$. For any $n\geqslant 1$ and $x\in B(z,r)$ we have

$$|R_n(S_D\varphi_k)(x)| = \left|\frac{1}{n!}\sum_{\beta\,:\,||\beta||=n}[D_x^\beta(S_D\varphi_k)(z+h(x-z))c_\beta\prod_{i=1}^d|x_i-z_i|^{\beta_i}]\right|,$$

where $h \in (0,1)$ depends on x, z and n, $c_\beta = ||\beta||!(\beta_1!\dots\beta_d!)^{-1}$. By (4.9) applied to the point $z + h(x-z) \in B(z,r)$ the above expression is bounded above by

$$c(D,k)\max(1,(\delta_D(z)/2)^{-2n-d-1})(d+3)^n \sum_{\beta\,:\,||\beta||=n} c_\beta \prod_{i=1}^{d} |x_i - z_i|^{\beta_i}. \tag{4.10}$$

But

$$\sum_{\beta\,:\,||\beta||=n} c_\beta \prod_{i=1}^{d} |x_i - z_i|^{\beta_i} = \left(\sum_{i=1}^{d} |x_i - z_i|\right)^n \leqslant \left(\left(d\sum_{i=1}^{d} |x_i - z_i|^2\right)^{1/2}\right)^n < d^{n/2} r^n.$$

Thus it is clear that for sufficiently small $r>0$, (4.10) goes to 0 as n goes to ∞ and this completes the proof of the theorem. $\square$

For any domain $D \subset \mathbb{R}^d$, we set $D_+ = \{x \in D : x_1 > 0\}$ and $D_- = \{x \in D : x_1 < 0\}$. For each $x = (x_1, x_2, \dots, x_d)$ we put $\widehat{x} = (-x_1, x_2, \dots, x_d)$. We say that D is symmetric relative to the x_1-axis if $\widehat{x} \in D$ whenever $x \in D$. If $D \subset \mathbb{R}^d$ is a connected bounded Lipschitz domain which is symmetric relative to the x_1-axis, it is easy to show that there exists an eigenfunction ψ_* with corresponding eigenvalue μ_* for the Dirichlet Laplacian, which is antisymmetric relative to the x_1-axis ($\psi_*(x) = -\psi_*(\widehat{x})$, $x \in D$) and (up to a sign) $\psi_*(x) > 0$ for $x \in D_+$ and $\psi_*(x) < 0$ for $x \in D_-$. We wish to prove a similar result for the Cauchy process.

Theorem 4.3. *Let $D \subset \mathbb{R}^d$ be a connected, bounded Lipschitz domain which is symmetric relative to the x_1-axis. Then there exists an eigenfunction φ_* for the Cauchy process with corresponding eigenvalue λ_* which is antisymmetric relative to the x_1-axis ($\varphi_*(x) = -\varphi_*(\widehat{x})$, $x \in D$) and (up to a sign) $\varphi_*(x) > 0$ for $x \in D_+$ and $\varphi_*(x) < 0$ for $x \in D_-$. Moreover, if φ is any eigenfunction with eigenvalue λ such that φ is antisymmetric relative to the x_1-axis and φ is different from φ_* ($\varphi \notin \mathrm{Span}\{\varphi_*\}$) then $\lambda_* < \lambda$. In other words, φ_* has the smallest eigenvalue among all eigenfunctions which are antisymmetric relative to x_1-axis.*

We first need some lemmas. Let $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_1 > 0\}$. For any $x, y \in \mathbb{R}^d_+$, $t > 0$, put $\tilde{p}(t,x,y) = p(t,x,y) - p(t,x,\widehat{y})$. It is easy to check that $\tilde{p}(t,x,y) > 0$. We wish to prove a similar result for the killed process.

Lemma 4.4. *Let D be as in Theorem 4.3. Fix $t > 0$ and let $0 < t_1 < t_2 < \dots < t_n < t$. For $x \in D_+$, $y \in D$ define $R[t_1](x,y) = p(t_1,x,y)$. Then*

$$R[t_1](x,y) - R[t_1](x,\widehat{y}) = \tilde{p}(t_1,x,y).$$

For $n \geqslant 2$, $x \in D_+$, $y \in D$, *define*

$$R[t_1, \dots, t_n](x, y) = \int_D R[t_1, \dots, t_{n-1}](x, z) p(t_n - t_{n-1}, z, y)\, dz$$
$$= \int_D \dots \int_D p(t_1, x, z_1) \dots p(t_n - t_{n-1}, z_{n-1}, y)\, dz_1 \dots dz_{n-1}.$$

Then for any $x, y \in D_+$,

$$R[t_1, \dots, t_n](x, y) - R[t_1, \dots, t_n](x, \widehat{y})$$
$$= \int_{D_+} \dots \int_{D_+} \tilde{p}(t_1, x, z_1) \dots \tilde{p}(t_n - t_{n-1}, z_{n-1}, y)\, dz_1 \dots dz_{n-1}. \tag{4.11}$$

Proof. For notational convenience, we will abbreviate $R[t_1, \dots, t_n](x, y)$ to $R_n(x, y)$. We will show (4.11) by induction. The case $n = 1$ is obvious. Assume (4.11) holds for n. For any $x, y \in D_+$ we have

$$R_{n+1}(x, y) - R_{n+1}(x, \widehat{y})$$
$$= \int_{D_+} R_n(x, z)(p(t_{n+1} - t_n, z, y) - p(t_{n+1} - t_n, z, \widehat{y}))\, dz$$
$$+ \int_{D_-} R_n(x, z)(p(t_{n+1} - t_n, z, y) - p(t_{n+1} - t_n, z, \widehat{y}))\, dz. \tag{4.12}$$

The integral in (4.12) equals

$$\int_{D_+} R_n(x, \widehat{z})(p(t_{n+1} - t_n, \widehat{z}, y) - p(t_{n+1} - t_n, \widehat{z}, \widehat{y}))\, dz.$$

Note that for any $s > 0$, $z, y \in \mathbb{R}^d$ we have $p(s, \widehat{z}, y) = p(s, z, \widehat{y})$ and $p(s, \widehat{z}, \widehat{y}) = p(s, z, y)$. Consequently the integral in (4.12) equals

$$\int_{D_+} R_n(x, \widehat{z})(p(t_{n+1} - t_n, z, \widehat{y}) - p(t_{n+1} - t_n, z, y))\, dz.$$

It follows that for any $x, y \in D_+$

$$R_{n+1}(x, y) - R_{n+1}(x, \widehat{y})$$
$$= \int_{D_+} (R_n(x, z) - R_n(x, \widehat{z}))(p(t_{n+1} - t_n, z, y) - p(t_{n+1} - t_n, z, \widehat{y}))\, dz.$$

Now (4.11) for $n + 1$ follows from (4.11) for n. $\square$

Lemma 4.5. *For any* $x, y \in D_+$, $t > 0$, *put* $\tilde{p}_D(t, x, y) = p_D(t, x, y) - p_D(t, x, \widehat{y})$. *Then we have* $\tilde{p}_D(t, x, y) > 0$.

Proof. We first note that for a bounded Lipschitz domain $P^x\{X_{\tau_D} \in \partial D\} = 0$, for $x \in D$, as seen from [9], Lemma 6. Let $x \in D_+$, $t > 0$. For any $B(y_0, r) \subset D_+$ we have

$$\begin{aligned}
\int_{B(y_0,r)} p_D(t,x,y)\,dy \\
&= P^x(X_t \in B(y_0,r); t < \tau_D) \\
&= \lim_{n\to\infty} P^x(X_{jt/n} \in D,\ j = 1,2,\dots,n,\ X_t \in B(y_0,r)) \\
&= \lim_{n\to\infty} \int_{B(y_0,r)} R[t/n,\dots,(n-1)t/n,t](x,y)\,dy. \qquad (4.13)
\end{aligned}$$

Similarly,

$$\int_{B(y_0,r)} p_D(t,x,\widehat{y})\,dy = \lim_{n\to\infty} \int_{B(y_0,r)} R[t/n,\dots,(n-1)t/n,t](x,\widehat{y})\,dy.$$

Therefore using Lemma 4.4 we obtain that $\int_{B(y_0,r)} \tilde{p}_D(t,x,y)\,dy$ equals

$$\lim_{n\to\infty} \int_{B(y_0,r)} \int_{D_+} \dots \int_{D_+} \tilde{p}(t/n,x,z_1)\dots\tilde{p}(t/n,z_{n-1},y)\,dz_1\dots dz_{n-1}\,dy.$$

The function $y \to \tilde{p}_D(t,x,y)$ is continuous on D because $y \to p_D(t,x,y)$ is continuous. Since $\tilde{p}(t,x,y) > 0$ it follows that $\tilde{p}_D(t,x,y) \geqslant 0$ for all $x, y \in D_+$, $t > 0$. However, to show that $\tilde{p}_D(t,x,y)$ is strictly positive requires additional work. To do this we will use the fact that the Cauchy kernel may be represented as the subordination of the Gaussian kernel $p^2(t,x,y)$ which, to avoid confusion here, we shall denote by

$$g(t,x,y) = (4\pi t)^{-d/2} \exp(-|x-y|^2/4t),$$

by the one-sided stable transition function

$$f_t(s) = \pi^{-1/2} t s^{-3/2} \exp(-t^2/4s) 1_{[0,\infty)}(s), \quad t > 0,\ s \in \mathbb{R}$$

of index 1/2. That is, we have

$$p(t,x,y) = \int_0^\infty g(s,x,y) f_t(s)\,ds,$$

$x, y \in \mathbb{R}^d$, $t > 0$. Similarly we have

$$\tilde{p}(t,x,y) = \int_0^\infty \tilde{g}(s,x,y) f_t(s)\,ds,$$

$x, y \in \mathbb{R}^d_+$, $t > 0$, where $\tilde{g}(s,x,y) = g(s,x,y) - g(s,x,\widehat{y})$.

Let us now denote by $g_\Omega(t,x,y)$, $x,y \in \Omega$, $t>0$ the heat kernel of the Brownian motion Y_t (running at twice the speed with kernel $g(t,x,y)$) killed on $\Omega \subset \mathbb{R}^d$. It is trivial that for $\Omega_1 \subset \Omega_2$ we have $g_{\Omega_1}(t,x,y) \leqslant g_{\Omega_2}(t,x,y)$, $x,y \in \Omega_1$, $t>0$. In addition, one easily checks that $\tilde{g}(t,x,y) = g_{\mathbb{R}^d_+}(t,x,y)$ and in particular we have $g_{D_+}(t,x,y) \leqslant g_{\mathbb{R}^d_+}(t,x,y)$ and $g_{D_+}(t,x,y) > 0$, $x,y \in D_+$, $t>0$. Let us put

$$u_{D_+}(t,x,y) = \int_0^\infty g_{D_+}(s,x,y) f_t(s)\, ds.$$

Then by standard arguments $u_{D_+}(t,x,y)$ satisfies the semigroup property. That is,

$$u_{D_+}(t_1+t_2,x,y) = \int_D u_{D_+}(t_1,x,z) u_{D_+}(t_2,z,y)\, dz, \tag{4.14}$$

for $x,y \in D_+$, $t_1,t_2 > 0$. Indeed,

$$\int_{D_+} \int_0^\infty g_{D_+}(s_1,x,z) f_{t_1}(s_1)\, ds_1 \int_0^\infty g_{D_+}(s_2,z,y) f_{t_2}(s_2)\, ds_2\, dz$$

$$= \int_0^\infty \int_0^\infty g_{D_+}(s_1+s_2,x,y) f_{t_1}(s_1)\, ds_1 f_{t_2}(s_2)\, ds_2.$$

Substituting $s = s_1 + s_2$, $ds = ds_1$ the previous expression equals

$$\int_0^\infty \int_0^\infty g_{D_+}(s,x,y) 1_{[s_2,\infty)}(s) f_{t_1}(s-s_2)\, ds f_{t_2}(s_2)\, ds_2$$

$$= \int_0^\infty g_{D_+}(s,x,y) \int_0^s f_{t_1}(s-s_2) f_{t_2}(s_2)\, ds_2\, ds.$$

But it is well known [8, p. 19] that $f_{t_1} * f_{t_2}(s) = f_{t_1+t_2}(s)$. This gives (4.14). It also follows easily that for fixed $t>0$, $x \in D_+$, the function $y \to u_{D_+}(t,x,y)$ is continuous on D_+ and $u_{D_+}(t,x,y) > 0$ for $x,y \in D_+$, $t>0$.

Now, for $x,y \in D_+$ we have

$$\tilde{p}(t,x,y) = \int_0^\infty g_{\mathbb{R}^d_+}(s,x,y) f_t(s)\, ds$$

$$\geqslant \int_0^\infty g_{D_+}(s,x,y) f_t(s)\, ds = u_{D_+}(t,x,y).$$

It follows that for $B(y_0, r) \subset D_+$, $x \in D_+$, $t > 0$,

$$\begin{aligned}
&\int_{B(y_0,r)} \tilde{p}_D(t,x,y)\,dy \\
&\quad = \lim_{n\to\infty} \int_{B(y_0,r)} \int_{D_+} \dots \int_{D_+} \tilde{p}(t/n,x,z_1)\dots\tilde{p}(t/n,z_{n-1},y)\,dz_1\dots dz_{n-1}\,dy \\
&\quad \geqslant \lim_{n\to\infty} \int_{B(y_0,r)} \int_{D_+} \dots \int_{D_+} u_{D_+}(t/n,x,z_1)\dots u_{D_+}(t/n,z_{n-1},y)\,dz_1\dots dz_{n-1}\,dy \\
&\quad = \int_{B(y_0,r)} u_{D_+}(t,x,y)\,dy,
\end{aligned}$$

where we used the semigroup property for u_{D_+} in the last equality. From this we see that

$$\tilde{p}_D(t,x,y) \geqslant u_{D_+}(t,x,y) > 0,$$

for all $x, y \in D_+$, $t > 0$, and this completes the proof of the lemma. □

Lemma 4.6. *For $x, y \in D$ we have $p_D(t,\widehat{x},y) = p_D(t,x,\widehat{y})$ and $p_D(t,\widehat{x},\widehat{y}) = p_D(t,x,y)$.*

Proof. This follows immediately from formula (4.13), the corresponding properties for $p(t,x,y)$ and the fact that D is symmetric relative to the x_1-axis. □

Proof of Theorem 4.3. It is easy to check that the kernel $\tilde{p}_D(t,x,y)$ defines a semigroup on $L^2(D_+)$. Let us denote this semigroup by $\tilde{P}_t^D$. By the general theory of and the strict positivity of $\tilde{p}_D(t,x,y)$ [26] there exists an orthonormal basis of eigenfunctions $\{\Psi_n\}$ for $L^2(D_+)$ and corresponding eigenvalues $\{a_n\}$ satisfying $0 < a_1 < a_2 \leqslant a_3 \leqslant \cdots$ for this operator. That is, $\tilde{P}_t^D \Psi_n(x) = \exp(-a_n t)\Psi_n(x)$, $x \in D_+$, $t > 0$, $n \in \mathbb{N}$. All the eigenfunctions Ψ_n are bounded and continuous by the properties of the kernel $\tilde{p}_D(t,x,y)$. In addition, strict positivity of $\tilde{p}_D(t,x,y)$ implies that the first eigenfunction Ψ_1 is strictly positive and a_1 is simple. All this follows from the general theory of heat semigroups as in [26].

Now define $\bar{\Psi}_n(x) = \Psi_n(x)$ for $x \in D_+$, $\bar{\Psi}_n(x) = -\Psi_n(\hat{x})$ for $x \in D_-$ and $\bar{\Psi}_n(x) = 0$ for $x \in D \backslash (D_+ \cup D_-)$. It is easy to check that $\bar{\Psi}_n$ is an eigenfunction for P_t^D with corresponding eigenvalue a_n. That is,

$$P_t^D \bar{\Psi}_n(x) = \exp(-a_n t)\bar{\Psi}_n(x), \quad x \in D, \quad t > 0, \quad n \in \mathbb{N}.$$

In fact using Lemma 4.6 we get for $x \in D_-$, $t > 0$, $n \in \mathbb{N}$

$$\begin{aligned}
e^{-a_n t}\bar{\Psi}_n(x) &= -\,e^{-a_n t}\Psi_n(\widehat{x}) = -\tilde{P}_t^D \Psi_n(\widehat{x}) \\
&\qquad = -\int_{D_+} p_D(t,\widehat{x},y)\Psi_n(y)\,dy + \int_{D_+} p_D(t,\widehat{x},\widehat{y})\Psi_n(y)\,dy
\end{aligned}$$

$$= -\int_{D_+} p_D(t,x,\hat{y})\Psi_n(y)\,dy + \int_{D_+} p_D(t,x,y)\Psi_n(y)\,dy$$
$$= \int_{D_-} p_D(t,x,y)\bar{\Psi}_n(y)\,dy + \int_{D_+} p_D(t,x,y)\bar{\Psi}_n(y)\,dy = P_t^D\bar{\Psi}_n(x).$$

For $x \in D \backslash (D_+ \cup D_-)$ and $x \in D_+$, this property may be checked similarly. Of course, a priori we do not know that $\bar{\Psi}_n$ is continuous for $x \in D \backslash (D_+ \cup D_-)$. Since all eigenfunctions for P_t^D have continuous extensions and Ψ_n is continuous on D_+, we must have that $\bar{\Psi}_n$ (defined as above) is continuous on D.

Analogously if φ_n is an eigenfunction which is antisymmetric relative to the x_1-axis ($\varphi_n(x) = \varphi_n(\hat{x})$) then we can show that $\tilde{\varphi}_n(x) = 1_{D_+}(x)\varphi_n(x)$, $x \in D_+$ is an eigenfunction for $\tilde{P}_t^D$ with corresponding eigenvalue λ_n. Thus there is one-to-one correspondence between the eigenfunctions for $\tilde{P}_t^D$ and the antisymmetric eigenfunctions for P_t^D. It follows that $\varphi_* = \bar{\Psi}_1$ and $\lambda_* = a_1$ and all the properties of φ_* and λ_* follow from the corresponding properties for Ψ_1 and a_1. This completes proof of the theorem. □

Theorem 4.7. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain and set*

$$\Delta_x = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

(i) *For $x \in D$ we have $\Delta_x \varphi_1(x) \leqslant 0$.*
(ii) *If in addition D is connected and symmetric relative to the x_1-taxis, then for $x \in D_+$ we have $\Delta_x \varphi_*(x) \leqslant 0$.*

Proof. Using Proposition 3.2(iii) twice we obtain

$$\frac{\partial^2 u_1}{\partial t^2}(x,t) = \lambda_1^2 u_1(x,t) - \lambda_1 P_t r_1(x) + \frac{\partial}{\partial t}(P_t r_1(x)), \quad (x,t) \in H_+.$$

Since u_1 is harmonic in H_+,

$$\frac{\partial^2 u_1}{\partial t^2}(x,t) = -\Delta_x u_1(x,t), \quad (x,t) \in H_+.$$

It follows that

$$\Delta_x u_1(x,t) = -\lambda_1^2 u_1(x,t) + \lambda_1 P_t r_1(x) - \frac{\partial}{\partial t}(P_t r_1(x)), \quad (x,t) \in H_+. \tag{4.15}$$

We know that for $x \in D$ we have $u_1(x,t) \to \varphi_1(x) > 0$, as $t \to 0^+$. Since $r_1 \in L^1(\mathbb{R}^d)$ and equals 0 on D, we get that $P_t r_1(x) \to 0$ for $x \in D$, as $t \to 0^+$. To finish the proof of the first part of the theorem we must show that for $x \in D$, $\lim_{t\to 0^+}(\frac{\partial}{\partial t}P_t r_1(x))$ exists and is non-negative and that $\Delta_x u_1(x,t)$ tends to $\Delta_x \varphi_1(x)$ as $t\to 0^+$. To do this observe that for $x \in D$ and $t>0$ we have

$$\frac{\partial(P_t r_1)}{\partial t}(x) = \int_{\mathbb{R}^d} \frac{\partial}{\partial t} p(t,x,y) r_1(y)\, dy.$$

Using the formula for $\frac{\partial}{\partial t}p(t,x,y)$ (Lemma 3.3(b)) and the fact that $r_1 \in L^1(\mathbb{R}^d)$ and $\mathrm{supp}(r_1) \subset D^c$, we obtain that for $x \in D$

$$\lim_{t\to 0^+} \frac{\partial(P_t r_1)}{\partial t}(x) = c_d \int_{\mathbb{R}^d} \frac{r_1(y)}{|x-y|^{d+1}}\, dy. \tag{4.16}$$

Since $\varphi_1(x)>0$ on D, $r_1(x) \geqslant 0$ on $\mathbb{R}^d$ so the limit in (4.16) is non-negative.

Now we will show that for $x \in D$, $\Delta_x u_1(x,t)$ tends to $\Delta_x \varphi_1(x)$ as $t \to 0^+$. Fix $x \in D$, $z \in D$ and $r>0$ such that $B(z,3r)\subset D$ and $x \in B(z,r)$. Let $f \in C^\infty(\mathbb{R}^d)$ be such that $f \equiv 1$ on $B(z,r)$ and $f \equiv 0$ on $B^c(z,2r)$. Set $g(y) = 1 - f(y)$, $y \in \mathbb{R}^d$. Recall that $p(t,x,y) = p(t,x-y)$ (where $p(t,y) = p(t,0,y)$). For any $t>0$,

$$\begin{aligned} u_1(x,t) &= \int_{\mathbb{R}^d} p(t,x-y)(\varphi_1 f)(y)\, dy + \int_{\mathbb{R}^d} p(t,x-y)(\varphi_1 g)(y)\, dy \\ &= \int_{\mathbb{R}^d} p(t,y)(\varphi_1 f)(x-y)\, dy + \int_{\mathbb{R}^d} p(t,x-y)(\varphi_1 g)(y)\, dy. \end{aligned} \tag{4.17}$$

Note that $\varphi_1 f = 0$ on $B^c(z,2r)$. Since φ_1 is real analytic on D it follows that $\varphi_1 f$ is C^∞ on $\mathbb{R}^d$. Therefore for any $t>0$ we have

$$\Delta_x\left(\int_{\mathbb{R}^d} p(t,y)(\varphi_1 f)(x-y)\, dy\right) = \int_{\mathbb{R}^d} p(t,y)\Delta_x(\varphi_1 f)(x-y)\, dy.$$

However, the last integral tends to $\Delta_x(\varphi_1 f)(x) = \Delta_x\varphi_1(x)$, as $t\to 0^+$.

On the other hand,

$$\Delta_x\left(\int_{\mathbb{R}^d} p(t,x-y)(\varphi_1 g)(y)\, dy\right) = \int_{\mathbb{R}^d} \Delta_x p(t,x-y)(\varphi_1 g)(y)\, dy. \tag{4.18}$$

Note that $\varphi_1 g \equiv 0$ on $B(z,r)$ and that $\varphi_1 g$ is bounded and has compact support. Using the formula for $\Delta_x p(t,x-y)$ (Lemma 3.3(c)) it is easy to show that the integral on the right-hand side of (4.18) tends to 0 as $t\to 0^+$. Hence (4.17) gives that $\Delta_x u_1(x,t)$ tends to $\Delta_x\varphi_1(x)$ as $t\to 0^+$ which finishes the proof of the first part of the theorem.

In the exact same way, we get (4.15) with u_1 replaced by u_*, λ_1 by λ_* and r_1 by r_*, where $u_*(x,t) = P_t\varphi_*(x)$, $r_*(x) = \lim_{t\to 0^+}(u_*(x,t)/t)$ for $x \in \mathrm{int}(D^c)$ and $r_*(x) = 0$ for $x \in \bar{D}$. Notice that $r_*(x) = -r_*(\widehat{x})$ and $r_*(x) \geqslant 0$ for $x \in \mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_1 > 0\}$. It follows that for $x \in D_+$ the expression on the right-hand side of (4.16) with r_1 replaced by r_* is non-negative. The rest of the proof is the same as in (i). This proves (ii) and completes the proof. $\square$

Proposition 4.8. *Let $D \subset \mathbb{R}^d$ be a connected bounded Lipschitz domain which is symmetric relative to the x_1-axis. Then*

$$\lambda_* \leqslant \sqrt{\mu_*}.$$

Proof. We will first verify that

$$\lambda_* = \inf_{u \in \mathscr{F}_*} \int_H |\nabla u(x,t)|^2 \, dx \, dt, \tag{4.19}$$

where

$$\mathscr{F}_* = \{u \in \mathscr{F} : ||\tilde{u}||_2 = 1 \text{ and } \tilde{u} \text{ is antisymmetric relative to the } x_1\text{-axis}\}.$$

Assume that λ_* has multiplicity $m \geqslant 1$ and that it corresponds to $\lambda_k, \ldots \lambda_{k+m-1}$, for some $k \geqslant 2$. Then by Theorem 3.8,

$$\lambda_* = \lambda_k = \inf_{u \in \mathscr{F}_k} \int_H |\nabla u(x,t)|^2 \, dx, \tag{4.20}$$

where

$$\mathscr{F}_k = \{u \in \mathscr{F} : \tilde{u} \perp \varphi_1, \ldots, \varphi_{k-1}; \ ||\tilde{u}||_2 = 1\}.$$

We will show that $\varphi_1, \ldots, \varphi_{k-1}$ are all symmetric relative to the x_1-axis. Toward this goal, let $l \in \{1, 2, \ldots, k-1\}$. Set $\psi(x) = \varphi_l(x) - \varphi_l(-x)$. If φ_l is not symmetric relative to the x_1-axis, then ψ is non-zero. By the symmetry of $p_D(t,x,y)$ we get that ψ is a non-zero eigenfunction corresponding to λ_l and of course it is antisymmetric relative to the x_1-axis. This however cannot occur due to Theorem 4.3. By the definition of $\mathscr{F}_*$ and $\mathscr{F}_k$ we see that $\mathscr{F}_* \subset \mathscr{F}_k$ and it follows from (4.20) that

$$\lambda_* = \inf_{u \in \mathscr{F}_k} \int_H |\nabla u(x,t)|^2 \, dx \leqslant \inf_{u \in \mathscr{F}_*} \int_H |\nabla u(x,t)|^2 \, dx,$$

which proves the right-hand side of (4.19).

To prove the other side of (4.19), set $u_*(x,t) = P_t \varphi_*(x)$, $(x,t) \in H_+$ and $u_*(x,0) = \varphi_*(x)$. Then $u_* \in \mathscr{F}_*$. By Proposition 3.6,

$$\lambda_* = \int_H |\nabla u_*(x,t)|^2 \, dx \geqslant \inf_{u \in \mathscr{F}_*} \int_H |\nabla u(x,t)|^2 \, dx,$$

and this proves the other half of (4.19).

Put $v_*(x,t) = \exp(-\sqrt{\mu_*}t)\psi_*(x)$, $(x,t) \in H$. By direct calculations as in the proof of Theorem 3.14, we get

$$\int_H |\nabla v_*(x,t)|^2 \, dx\, dt = \sqrt{\mu_*}.$$

It is easy to check that $v_* \in \mathscr{F}_*$ and from this it follows that

$$\lambda_* \leqslant \int_H |\nabla v_*(x,t)|^2 \, dx\, dt = \sqrt{\mu_*},$$

proving the proposition. □

5. The one-dimensional case

In this section we study the Cauchy eigenvalue problem in one dimension for the set $D = (-1,1)$. In such a simple case we will be able to prove several detailed properties for the eigenfunctions functions φ_n similar to those discussed in the introduction for the eigenfunctions of the Laplacian. Our main result in this section is Theorem 5.3 which provides information on the second eigenfunction.

First, let us recall that the first eigenvalue λ_1 is simple, its corresponding eigenvalue φ_1 is positive (true for any domain of finite volume in any dimension) and that by Corollary 2.2, $1 < \lambda_1 < 3\pi/8$. In addition to this, we also have the following additional information on the shape of φ_1.

Theorem 5.1. *Let* $D = (-1,1)$. *Then* φ_1 *is symmetric relative to the origin and concave. It is non-decreasing on* $(-1,0)$ *and non-increasing on* $(0,1)$.

Proof. Put $\widehat{\varphi}_1(x) = \varphi_1(x) + \varphi_1(-x)$. It is easy to show that $\widehat{\varphi}_1$ is also an eigenfunction corresponding to λ_1. Since λ_1 has multiplicity 1 it follows that φ_1 is symmetric. The concavity of φ_1 follows from Theorem 4.7. Since φ_1 is symmetric and concave on $(-1,1)$ it must be non-decreasing on $(-1,0)$ and non-increasing on $(0,1)$ and this completes the proof. □

For our next result we need to recall a few facts for the one-dimensional Cauchy process. In one dimension the transition densities are given by

$$p(t,x,y) = \frac{1}{\pi} \frac{t}{t^2 + (x-y)^2}, \tag{5.1}$$

where $t > 0$, $x, y \in \mathbb{R}$. For any $a > 0$ and $x \in (-a,a)$ we have, by (2.12), that

$$E^x(\tau_{(-a,a)}) = \sqrt{a^2 - x^2}. \tag{5.2}$$

The distribution of $X(\tau_{(a,b)})$ is given in [7] explicitly by the formula

$$P^x(X(\tau_{(a,b)})\in B) = \frac{1}{\pi}\int_B \frac{(r^2-|x-x_0|^2)^{1/2}}{(|y-x_0|^2-r^2)^{1/2}|x-y|}\,dy, \tag{5.3}$$

where $a, b\in\mathbb{R}$, $a<b$, $x\in(a,b)$, $B\subset(a,b)^c$, $x_0=(a+b)/2$, $r=(b-a)/2$.

Lemma 5.2. (i) *For any* $a\in(0,1)$ *and* $x\in(-1,-a)$,

$$P^x(X(\tau_{(-1,-a)})\in(a,1))\leqslant\frac{(1-a)^2}{8\pi a^2}.$$

(ii) *For any* $a\in[1/4,1)$ *and* $x\in(-1,-a)\cup(a,1)$,

$$\frac{1}{E^x(\tau_{(-1,-a)\cup(a,1)})}\geqslant g(a),$$

where

$$g(a)=\frac{2}{1-a}\left(1-\frac{(1-a)^2}{8\pi a^2}\right)=\frac{8\pi a^2-(1-a)^2}{4\pi a^2(1-a)}.$$

The function g *is positive and increasing on* $[1/4,1)$.

Proof. By (5.3) we get

$$P^x(X(\tau_{(-1,-a)})\in(a,1)) = \frac{1}{\pi}\int_a^1 \frac{(r^2-|x-x_0|^2)^{1/2}}{(|y-x_0|^2-r^2)^{1/2}|x-y|}\,dy, \tag{5.4}$$

where $r=(1-a)/2$ and $x_0=(-1-a)/2$. For any $x\in(-1,-a)$ and $y\in(a,1)$ we have $r^2-|x-x_0|^2\leqslant r^2$, $|y-x_0|^2-r^2\geqslant(2a)^2$ and $|x-y|\geqslant 2a$. Therefore (i) follows from (5.4).

Let

$$q=\sup_{x\in(-1,-a)}E^x(\tau_{(-1,-a)})=\sup_{x\in(a,1)}E^x(\tau_{(a,1)})=(1-a)/2$$

and

$$\begin{aligned}p=\sup_{x\in(-1,-a)}P^x(X(\tau_{(-1,-a)})\in(a,1))&=\sup_{x\in(a,1)}P^x(X(\tau_{(a,1)})\in(-a,-1))\\ &\leqslant(1-a)^2/(8\pi a^2).\end{aligned}$$

By the strong Markov property we have that for any $x \in (-1,-a) \cup (a,1)$

$$E^x(\tau_{(-1,-a)\cup(a,1)}) \leqslant \sum_{k=0}^{\infty} qp^k = \frac{q}{1-p}.$$

Note also that for $a \in [1/4, 1)$ we have $(1-a)^2/(8\pi a^2) < 1$ (the interval $[1/4, 1)$ is not of course optimal). Therefore the estimate for $E^x(\tau_{(-1,-a)\cup(a,1)})$ follows from the above bounds for p and q, and this proves (ii). $\square$

Theorem 5.3. *Let $D = (-1, 1)$. Then $2 \leqslant \lambda_2 \leqslant \pi$, it has multiplicity 1, its eigenfunction φ_2 is negative on $(-1, 0)$, positive on $(0, 1)$ and antisymmetric relative to the origin. φ_2 is convex on $(-1, 0)$ and concave on $(0, 1)$. In particular, there is an $a \in (0, 1)$ such that φ_2 is non-decreasing on $(-a, a)$ and non-increasing on $(-1, -a)$ and $(a, 1)$. The antisymmetric property is inherited by u_2 in the sense that $u_2(x, t) = -u_2(-x, t)$, $(x, t) \in H$. In addition, u_2 has two nodal parts* (see Fig. 1).

$$A = \{(x,t) \in H : x < 0,\ t > 0\} \cup \{(x,0) \in H : x \in (-1,0)\}$$

and

$$B = \{(x,t) \in H : x > 0,\ t > 0\} \cup \{(x,0) \in H : x \in (0,1)\}.$$

with $u_2(x,t) < 0$ for $(x,t) \in A$ and $u_2(x,t) > 0$ for $(x,t) \in B$.

Remark 1. Of course, whenever φ_2 is an eigenfunction with eigenvalue λ_2 so is $-\varphi_2$, hence the above statements should be interpreted "up to sign."

Proof. By Theorem 4.3 there exists an eigenfunction φ_* with corresponding eigenvalue λ_* satisfying the following properties: $||\varphi_*||_2 = 1$, φ_* is antisymmetric, negative on $(-1, 0)$ and positive on $(0, 1)$. By Theorem 4.7(ii), φ_* is concave on $(0, 1)$ and convex on $(-1, 0)$. If we denote $u_*(x,t) = P_t\varphi_*(x)$, $(x,t) \in H_+$, then u_* has two nodal parts A and B such as in the formulation of the theorem.

By Proposition 4.8 we have $\lambda_* \leqslant \sqrt{\mu_2}$, where μ_2 is the second eigenvalue of the Dirichlet Laplacian in $D = (-1, 1)$. Hence $\lambda_* \leqslant \pi$. On the other hand, note that $\tilde{A} = (-1, 0)$ (recall that $\tilde{A} = \{x \in D : (x,0) \in A\}$). So by Proposition 3.12 and scaling we get

$$\lambda_* \geqslant \lambda_1((-1,0)) = 2\lambda_1((-1,1)) = 2\lambda_1 \geqslant 2.$$

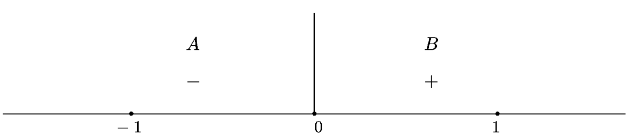

Fig. 1. Nodal parts for u_2.

We will now show that $\lambda_2 = \lambda_*$ and that φ_* is the unique eigenfunction corresponding to λ_2. Suppose on the contrary that there exists an eigenfunction φ (with $||\varphi||_2 = 1$) corresponding to λ_2 which is different from φ_*. That is, $\varphi \neq \varphi_*$ and $\varphi \neq -\varphi_*$. By Theorem 4.1 $\varphi(x)$ is real analytic on D. In particular, the zeros of $\varphi(x)$ have no accumulation points in D. We will say that a real analytic function $f : D \to \mathbb{R}$ changes sign at the point $x_0 \in D$ if there exists $\varepsilon > 0$ such that $f(x) > 0$ on $(x_0, x_0 + \varepsilon)$ and $f(x) < 0$ on $(x_0 - \varepsilon, x_0)$ or such that $f(x) < 0$ on $(x_0, x_0 + \varepsilon)$ and $f(x) > 0$ on $(x_0 - \varepsilon, x_0)$.

Consider $v(x, t) = P_t\varphi(x)$, $(x, t) \in H_+$ and $v(x, 0) = \varphi(x)$, $x \in \mathbb{R}^d$. By Theorem 3.11 the function v has no more than two nodal parts. Since φ is orthogonal to φ_1 it must change its sign and hence v has exactly two nodal parts.

We will show that φ changes sign at no more than two points in D. Assume this is not the case. That is, φ changes sign at more than two points. We may also assume that $\varphi(x) \geqslant 0$ for $x \in (a_0, a_1)$ and $x \in (a_2, a_3)$ and $\varphi(x) \leqslant 0$ for $x \in (a_1, a_2)$ and $x \in (a_3, a_4)$, where

$$-1 \leqslant a_0 < a_1 < a_2 < a_3 < a_4 \leqslant 1.$$

Let us denote $A_i = \{(x, 0) \in H : x \in (a_{i-1}, a_i) \text{ and } \varphi(x) \neq 0\}$, $i = 1, 2, 3, 4$. All sets A_1, A_2, A_3, A_4 are non-empty. Let $P_+ = \{(x, t) \in H : v(x, t) > 0\}$ and $P_- = \{(x, t) \in H : v(x, t) < 0\}$. Of course $A_1 \cup A_3$ belongs to P_+.

If P_+ is connected then P_- is not connected and v has more than two nodal parts (see Fig. 2). If P_+ is not connected then again v has more than two parts. This gives a contradiction and shows that φ changes its sign at no more than two points in D.

By Theorem 4.3, φ is not antisymmetric. This follows from the fact that $\varphi \neq \varphi_*$, $\varphi \neq -\varphi_*$ and that every antisymmetric eigenfunction (with norm 1) different from φ_* (or $-\varphi_*$) has greater corresponding eigenvalue. Set $\widehat{\varphi}(x) = \varphi(x) + \varphi(-x)$. By the last remark $\widehat{\varphi}$ is not identically zero. It easy to show that $\widehat{\varphi}$ is an eigenfunction with corresponding eigenvalue λ_2. Hence from our assumptions on φ, it follows that there exists a symmetric eigenfunction with corresponding eigenvalue λ_2. Therefore we assume that φ is symmetric. We have shown that $v = P_t\varphi$ has two nodal parts and that φ changes its sign at no more than two points in D. Since it must change sign and is symmetric, it must change sign at exactly two points in D. Therefore there is an $a \in (0, 1)$ such that $\varphi(a) = \varphi(-a) = 0$ and φ change sign at a and $-a$. We may assume that $\varphi(x) \geqslant 0$ for $x \in (-1, -a) \cup (a, 1)$ and $\varphi(x) \leqslant 0$ for $x \in (-a, a)$.

Let us denote

$$U_+ = \{(x, t) \in H : v(x, t) > 0\}$$

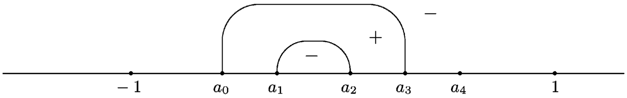

Fig. 2. Nodal parts for v.

and

$$U_- = \{(x,t) \in H : v(x,t) < 0\}.$$

If U_- is unbounded then U_+ would not be connected and v would have more than two nodal parts (see Fig. 3). Hence, U_- is bounded. Denote by $N = \{(x,t) \in H_+ : v(x,t) = 0\}$ the nodal line for v. Note that we have used H_+ (and not H) in the definition of N. The function φ is analytic in D, hence its zeros have no accumulation points in D. Thus the zeros at points $-a$ and a are isolated. If $\bar{N} \cap \{(x,0) : x \in \mathbb{R}\}$ is different from $\{-a,a\}$, then either U_+ or U_- would not be connected and v would have more than 2 nodal parts. We must therefore have $\bar{N} \cap \{(x,0) : x \in \mathbb{R}\} = \{-a,a\}$. Let

$$l_+ = \{(x,0) : x > 1\} \quad \text{and} \quad l_- = \{(x,0) : x < -1\}.$$

From the above it follows that for each point $(x,0) \in l_+ \cup l_-$ there exist $s = s(x) > 0$ such that for $(y,t) \in \{(y,t) \in H_+ : |(y,t) - (x,0)| < s\}$, we have $v(y,t) > 0$. Let $r(x) = \lim_{t \to 0^+} v(x,t)/t$ for $|x| > 1$ and 0 for $|x| \leqslant 1$. This $r(x)$ is nothing more than the function defined in Proposition 3.15 and it follows that $r(x) \geqslant 0$ for all $x \in \mathbb{R}$.

We shall now apply Proposition 3.15 which is the key argument in this proof. From this it follows that if for some $(x_0, t_0) \in H$ we have $v(x_0, t_0) \geqslant 0$, then for all $t > t_0$ we have $v(x_0, t) > 0$. Recall that for $x \in (-1, -a] \cup [a, 1)$ we have $\varphi(x) = v(x,0) \geqslant 0$ and for all $|x| \geqslant 1$ we have $\varphi(x) = v(x,0) = 0$. Therefore for all $|x| \geqslant a$ and $t \geqslant 0$ we get $v(x,t) \geqslant 0$. Thus $U_- \subset (-a,a) \times [0,\infty)$. Since the first eigenvalue μ_1 for the Dirichlet Laplacian in $(-a,a)$ is $\pi^2/4a^2$, it follows from Theorem 3.18 that $\lambda_2 \geqslant \pi/2a$. On the other hand, for the interval $(-1,1)$ the second eigenvalue of the Dirichlet Laplacian is π^2 and hence Theorem 3.14 gives $\lambda_2 \leqslant \pi$. We conclude that

$$\frac{\pi}{2a} \leqslant \lambda_2 \leqslant \pi.$$

From this it follows that $a \geqslant 1/2$. Since $\varphi(x) \leqslant 0$ for all $x \in (-a,a)$, it follows that $\tilde{U}_+ \subset (-1,-a) \cup (a,1)$, where $\tilde{U}_+ = \{x \in D : (x,0) \in U_+\}$. The set $(-1,-a) \cup (a,1)$ is trivially a bounded Lipschitz domain. By Proposition 3.12, the fact that $a \geqslant 1/2$, and domain monotonicity of λ_1, we obtain

$$\lambda_2 \geqslant \lambda_1((-1,-a) \cup (a,1)) \geqslant \lambda_1((-1,-1/2) \cup (1/2,1)).$$

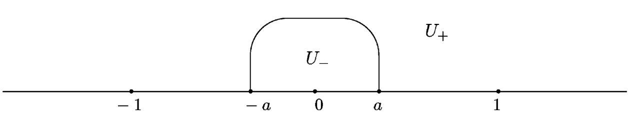

Fig. 3. Nodal parts for v.

Put $\Omega = (-1, -1/2) \cup (1/2, 1)$. From Proposition 2.1 and Lemma 5.2 we conclude that

$$\lambda_1(\Omega) \geqslant \frac{1}{\sup_{x \in \Omega} E^x(\tau_\Omega)} \geqslant \frac{2}{1 - 1/2}\left(1 - \frac{(1 - 1/2)^2}{8\pi(1/2)^2}\right) = 4\left(1 - \frac{1}{8\pi}\right).$$

Thus

$$\pi \geqslant \lambda_2 \geqslant \lambda_1(\Omega) \geqslant \left(\sup_{x \in \Omega} E^x(\tau_\Omega)\right)^{-1} \geqslant 4 - 1/(2\pi) > \pi,$$

which gives a contradiction and completes the proof of the theorem. □

When dealing with Brownian motion (the Laplacian) in $D = (-1, 1)$, we know the full set of eigenfunctions and eigenvalues. The above theorem gives some information, although not as precise, on the second eigenvalue and the second eigenfunction for the Cauchy process. It would be very desirable to develop some general techniques to study properties of higher eigenvalues and eigenfunctions for the Cauchy process and, in particular, to gain a better understanding of their geometric properties. For general eigenfunctions this goal seems too ambitious at this point. In the next theorem we aim to gain some understanding for φ_3 and for the corresponding Steklov function u_3. We hope that even some limited knowledge on φ_3 and u_3 can provide some intuition about other eigenfunctions.

Theorem 5.4. *Let $D = (-1, 1)$. Then, $3.4 \leqslant \lambda_3 \leqslant 3\pi/2$, it has multiplicity 1, its eigenfunction φ_3 is symmetric and has two zeros at $-a, a$, where $a \in [1/3, 0.6]$. Moreover, φ_3 is positive on $(-1, -a) \cup (a, 1)$ and negative on $(-a, a)$. The corresponding Steklov function u_3 satisfies $u_3(x, t) = u_3(-x, t)$, $(x, t) \in H$ and has two nodal parts A and B. Assume that $u_3(x, t) > 0$ for $(x, t) \in A$ and $u_3(x, t) < 0$ for $(x, t) \in B$. Then $B \subset (-a, a) \times [0, \infty)$ and B is bounded. If $u_3(x_0, t_0) \geqslant 0$ for some $(x_0, t_0) \in H$ then for all $t \geqslant t_0$ we have $u_3(x_0, t) > 0$.*

A remark similar to that after Theorem 5.3 applies here as well concerning the sign of φ_3.

Proof. By Theorem 3.11 u_3 has at most 3 nodal parts. In follows from the argument of Theorem 5.3 that if u_3 has 2 nodal parts then it changes its sign at no more than 2 points. In a similar way, it may be shown that if the function u_3 has 3 nodal parts then it changes sign at no more than 4 points. From this we conclude that if u_3 has 2 nodal parts then it changes sign at no more than 2 points and if u_3 has 3 nodal parts then it changes sign at no more then 4 points.

Next we show that φ_3 is symmetric. Assume on the contrary that φ_3 is not symmetric. Put $\varphi(x) = \varphi_3(x) - \varphi_3(-x)$, $x \in D$. Then φ is antisymmetric and orthogonal to the antisymmetric eigenfunction φ_2. Therefore φ must change sign at least at 3 points (including the origin 0). Let $v(x, t) = P_t\varphi(x)$, $(x, t) \in H$. Since φ is antisymmetric and changes sign at at least 3 points, v must have at least 4 nodal parts, which gives a contradiction. Therefore φ_3 is symmetric. From the above it follows that only the following cases can occur:

Case 1: u_3 has 2 nodal parts and φ_3 changes sign at 2 points.

Case 2: u_3 has 3 nodal parts and φ_3 changes sign at 2 points.

Case 3: u_3 has 3 nodal parts and φ_3 changes sign at 4 points.

Assume that Cases 2 and 3 do not occur. (These cases will be ruled out later, they represent the most difficult part of the proof.) Since we have not yet shown that λ_3 has multiplicity 1, our assumption is the following: For any eigenfunction corresponding to λ_3, Cases 2 and 3 cannot happen. Under this assumption we will show that the functions φ_3 and u_3 have the properties asserted by the theorem.

As in the proof of Theorem 5.3, there exists an $a \in (0,1)$ such that $\varphi_3(a) = \varphi_3(-a) = 0$ and φ_3 change sign at a and $-a$. We may assume that $\varphi_3(x) \geqslant 0$ for $x \in (-1,-a) \cup (a,1)$ and $\varphi_3(x) \leqslant 0$ for $x \in (-a,a)$. Put $A = \{(x,t) \in H : u_3(x,t) > 0\}$ and $B = \{(x,t) \in H : u_3(x,t) < 0\}$ (see Fig. 4). Let $r_3(x) = \lim_{t \to 0^+} u_3(x,t)$ for $|x| > 1$ and $r_3(x) = 0$ for $|x| \leqslant 1$. By the same arguments as in the proof of Theorem 5.3 we get that $r_3(x) \geqslant 0$ for all $x \in \mathbb{R}$. Proposition 3.15 now yields that if for some $(x_0, t_0) \in H$ we have $u_3(x_0, t_0) \geqslant 0$, then for all $t > t_0$ we have $u_3(x_0, t) > 0$. Note that for all $x \in (-1,-a) \cup (a,1)$ we have $\varphi_3(x) = u_3(x,0) \geqslant 0$ and for all $|x| \geqslant 1$ we have $u_3(x,0) = 0$. Therefore for all $|x| \geqslant a$ and $t \geqslant 0$ we have $u_3(x,t) \geqslant 0$. Hence $B \subset (-a,a) \times [0,\infty)$. Of course, B must be bounded or else, as before, A will not be connected.

From Proposition 3.17 we get that $\varphi_3(x) > 0$ for all $x \in (-1,-a) \cup (a,1)$. We will also show that $\varphi_3(x) < 0$ for all $x \in (-a,a)$. We know that $\varphi_3(x) \leqslant 0$ for all $x \in (-a,a)$. If $\varphi_3(x_0) = 0$ for some $x_0 \in (-a,a)$ then by Proposition 3.15, $u_3(x_0,t) > 0$ for all $t > 0$ and B will not be connected. Therefore the eigenfunction φ_3 on the set D has exactly 2 zeros at points $-a$ and a.

We next estimate λ_3 and a. Since $B \subset (-a,a) \times [0,\infty)$, Theorem 3.18 gives $\lambda_3 \geqslant \pi/(2a)$. On the other hand, by Theorem 3.14 we have $\lambda_3 \leqslant \sqrt{\mu_3} = \sqrt{(3\pi/2)^2} = 3\pi/2$ (as before, μ_3 is the solution of the Dirichlet eigenvalue problem (3.31)). Therefore $\pi/(2a) \leqslant 3\pi/2$ and so $a \geqslant 1/3$. This gives the desired upper bound for λ_3 and the lower bound for a.

We also have $\tilde{A} = (-1,-a) \cup (a,1)$ (recall that $\tilde{A} = \{x \in D : (x,0) \in A\}$) and by Proposition 3.12 we get

$$\lambda_3 \geqslant \lambda_1(\tilde{A}) \geqslant \left(\sup_{x \in \tilde{A}} E^x(\tau_{\tilde{A}}) \right)^{-1}.$$

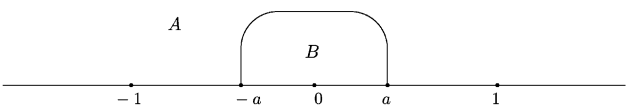

Fig. 4. Nodal parts for u_3.

Since $a \geqslant 1/3$, we obtain

$$\lambda_3 \geqslant \left(\sup_{x \in \tilde{A}} E^x(\tau_{\tilde{A}}) \right)^{-1} \geqslant g(a),$$

where g is given in Lemma 5.2.

Put $f(x) = \pi/(2x)$, $x \in [1/3, 1]$. We have

$$\lambda_3 \geqslant \max\{f(a), g(a)\}.$$

We note that f, g are continuous on $[1/3, 1)$, f is decreasing and g is increasing. We also have $f(1/3) = 3\pi/2$, $g(1/3) \leqslant 2/(1 - 1/3) = 3 < 3\pi/2$, $f(1) = \pi/2$ and $\lim_{b \to 1^-} g(b) = \infty$. Therefore there is exactly one $x_0 \in (1/3, 1)$ such that $f(x_0) = g(x_0)$. Moreover,

$$\lambda_3 \geqslant \max\{f(a), g(a)\} \geqslant f(x_0)$$

and for any $x \in (x_0, 1)$ we have $\lambda_3 \geqslant f(x_0) \geqslant f(x)$. We will show that $0.46 > x_0$. Indeed, $g(0.46) > 3.48$ and $f(0.46) < 3.42$. Hence $f(0.46) < g(0.46)$ which gives $0.46 > x_0$. Thus $\lambda_3 \geqslant f(0.46) \geqslant 3.4$.

We will show that $a \leqslant 0.6$. We have $3\pi/2 \geqslant \lambda_3 \geqslant g(a)$. Since g is increasing it suffices to show that $g(0.6) > 3\pi/2$. In fact, $g(0.6) > 4.8 > 3\pi/2$.

Next we prove that λ_3 has multiplicity 1. As before, we argue by contradiction. Assume there are two eigenfunctions Φ_1, Φ_2 corresponding to λ_3 which are orthogonal ($\int_D \Phi_1(x)\Phi_2(x)\,dx = 0$) and $||\Phi_1||_2 = ||\Phi_2||_2 = 1$. Let us recall that we are under the assumption that Cases 2 and 3 do not occur. Therefore both Φ_1 and Φ_2 satisfy the conditions of case 1. Hence the functions Φ_1 and Φ_2 are symmetric and have exactly 2 zeros at points $-a_i$, a_i. If $a_1 = a_2$ then Φ_1 and Φ_2 would not be orthogonal and this cannot happen.

We may assume that $0 < a_1 < a_2 < 1$ and that for $i = 1, 2$, we have $\Phi_i(x) > 0$ for $x \in (-1, -a_i) \cup (a_i, 1)$ and $\Phi_i(x) < 0$ for $x \in (-a_i, a_i)$. Consider functions $f_t = (1 - t)\Phi_2 - t\Phi_1$, $t \in [0, 1]$. Of course, for each $t \in [0, 1]$ the function f_t is also an eigenfunction corresponding to λ_3. Note that for $x \in [-a_2, -a_1] \cup [a_1, a_2]\, f_t(x) < 0$. Of course, $f_0 = \Phi_2$ and $f_0(x) < 0$ for $x \in [-a_1, a_1]$. Let

$$s = \inf\{t \in [0, 1] :\ \max\{f_t(x) : x \in [-a_1, a_1]\} = 0\}.$$

Since $f_1 = -\Phi_1$, $s < 1$. We have $\max\{f_s(x) : x \in [-a_1, a_1]\} = 0$. Let us denote the point at which this maximum is attained by y_1. We have $f_s(y_1) = 0$ and $f_s(x) \leqslant 0$ for all $x \in [-a_1, a_1]$. Since for $x \in [-a_2, -a_1] \cup [a_1, a_2]$ we have $f_s(x) < 0$ and f_s (as an eigenfunction for λ_3) is orthogonal to $\varphi_1 > 0$, there exists $y_2 \in (-1, -a_2) \cup (a_2, 1)$ such that $f_s(y_2) > 0$. Of course, f_s is symmetric so we may assume that $y_2 \in (a_2, 1)$ and $f_s(-y_2) = f_s(y_2) > 0$. Now for sufficiently small $\varepsilon > 0$ we have $f_{s+\varepsilon}(-y_2) = f_{s+\varepsilon}(y_2) > 0$, $f_{s+\varepsilon}(y_1) > 0$ and $f_{s+\varepsilon}(-a_2) = f_{s+\varepsilon}(a_2) < 0$ (because for all $t \in (0, 1]\, f_t(a_2) < 0$) . Since $-y_2 < -a_2 < y_1 < a_2 < y_2$ we obtain that f_s changes sign in more than at 2 points. But recalling again that we are under the assumption of

Case 1, this cannot happen. So, (under assumption that Cases 2 and 3 cannot occur) we obtain a contradiction, and conclude that λ_3 has multiplicity 1.

Our goal now is to show that for any eigenfunction corresponding to λ_3, Cases 2 and 3 cannot happen. To obtain a contradiction let us assume that there exists an eigenfunction corresponding to λ_3, denote it again by φ_3, such that φ_3 satisfies conditions of either Case 2 or Case 3. Recall that in both cases we may assume that φ_3 is symmetric. We look at each of the two cases separately.

Case 2: $u_3 = P_t\varphi_3$ has 3 nodal parts and φ_3 changes sign at 2 points. Let $a \in (0,1)$ and assume that φ_3 changes sign at $-a$, a and $\varphi_3(x) \geqslant 0$ for $x \in (-1,-a) \cup (a,1)$ and $\varphi_3(x) \leqslant 0$ for $x \in (-a,a)$. We will consider two subcases.

Case 2a: The set $\{(x,t) \in H : u(x,t) > 0\}$ is not connected.

Under this assumption, u_3 has 2 nodal parts A, B, on which $u_3 > 0$ and a nodal part C on which $u_3 < 0$, see Fig. 5.

Since the set $\{(x,t) \in H : u(x,t) > 0\} = A \cup B$ is not connected the nodal part $C = \{(x,t) \in H : u(x,t) < 0\}$ is not bounded and by the symmetry of φ_3, we must have $u_3(0,t) < 0$ for all $t > 0$. On the other hand, we know that φ_3 is orthogonal to φ_1 and so

$$\int_{-1}^{1} \varphi_3(x)\varphi_1(x)\,dx = 0.$$

Let us recall that φ_1 is positive, non-constant, symmetric, non-decreasing on $(-1,0)$ and non-increasing on $(0,1)$. Therefore we get

$$\int_{-1}^{1} \varphi_3(x)\varphi_1(a)\,dx > 0,$$

which implies

$$\int_{-1}^{1} \varphi_3(x)\,dx > 0.$$

But for $t > 0$,

$$u_3(0,t) = P_t\varphi_3(0) = \frac{1}{\pi}\int_{-1}^{1} \frac{t}{t^2+y^2}\,\varphi_3(y)\,dy.$$

It follows that

$$tu_3(0,t) \to \frac{1}{\pi}\int_{-1}^{1} \varphi_3(y)\,dy$$

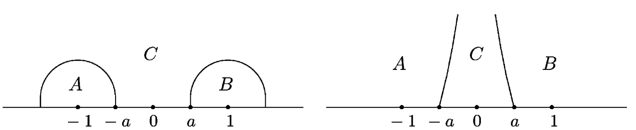

Fig. 5. Nodal parts for u_3. Two possibilities in Case 2a.

as $t\to\infty$. Therefore for sufficiently large t we have $u_3(0,t)>0$ which contradicts the fact that $u_3(0,t)<0$ for all $t>0$ and Case 2a cannot occur.

Case 2b: The set $\{(x,t)\in H: u_3(x,t)>0\}$ is connected.

Then u_3 has 2 nodal parts A and B on which $u_3<0$ and a nodal part C on which $u_3>0$, see Fig. 6.

As before, we have $r_3(x)\geqslant 0$ so (by Proposition 3.15) it follows that $A\subset(-a,0)\times[0,\infty)$. Therefore by Theorem 3.18, $\lambda_3\geqslant\pi/a$. Since $3\pi/2\geqslant\lambda_3$, we get $3\pi/2\geqslant\pi/a$. Hence, $a\geqslant 2/3$. Put $\Omega=(-1,-2/3)\cup(2/3,1)$ and recall that

$$\tilde{C}=\{x\in D:(x,0)\in C\}\subset(-1,-a)\cup(a,1).$$

By Proposition 3.12 and domain monotonicity of λ_1, we have

$$\begin{aligned}\lambda_3&\geqslant\lambda_1((-1,-a)\cup(a,1))\geqslant\lambda_1((-1,-2/3)\cup(2/3,1))\\&\geqslant\left(\sup_{x\in\Omega}E^x(\tau_\Omega)\right)^{-1}\geqslant g(2/3)>5>3\pi/2,\end{aligned}$$

which gives a contradiction (g is given in Lemma 5.2).

It remains to rule out Case 3.

Case 3: u_3 has 3 nodal parts and φ_3 changes sign at 4 points. Let $a,b\in(0,1)$, $a<b$ and assume that φ_3 changes sign at $-b$, $-a$, a, b and that $\varphi_3(x)\geqslant 0$ for $x\in(-1,-b)\cup(-a,a)\cup(b,1)$ and $\varphi_3(x)\leqslant 0$ for $x\in(-b,-a)\cup(a,b)$. As above, we will consider 2 subcases.

Case 3a: The set $\{x,t)\in H: u_3(x,t)<0\}$ is connected (see Fig. 7).

Under this assumption, u_3 has 2 nodal parts A, B on which $u_3>0$ and a nodal part C on which $u_3<0$. Note that $r_3(x)\geqslant 0$ for all $x\in\mathbb{R}$ and therefore by Proposition 3.15 we obtain that $u_3(x,t)$ has the following property: If $u_3(x_0,t_0)\geqslant 0$ for some $(x_0,t_0)\in H$ then for all $t>t_0$, $u_3(x_0,t)>0$. This contradicts the connectedness of the set $\{(x,t)\in H: u_3(x,t)<0\}$.

Case 3b: The set $\{(x,t)\in H: u_3(x,t)<0\}$ is not connected (see Fig. 8).

This is the most difficult case to rule out. In this case u_3 has 2 nodal parts A, B on which $u_3<0$ and a nodal part C on which $u_3>0$. Note that $r_3(x)\geqslant 0$ for all $x\in\mathbb{R}$. Therefore by Proposition 3.15, $A\subset(-b,-a)\times[0,\infty)$ and $B\subset(a,b)\times[0,\infty)$. By Proposition 3.17 we get that $\varphi_3(x)>0$ for $x\in(-1,-b)\cup(b,1)$. By Theorem 3.14 we get $\lambda_3\geqslant\pi/(b-a)$ and by Theorem 3.14, $\lambda_3\leqslant 3\pi/2$. Therefore $b-a\geqslant 2/3$. Put $\Omega=(-1,-b)$. Then $\tau_\Omega\leqslant\tau_D$ and by (2.2) for $x\in\Omega$ we have

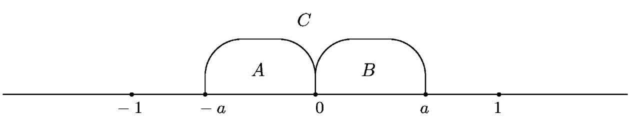

Fig. 6. Nodal parts for u_3. Case 2b.

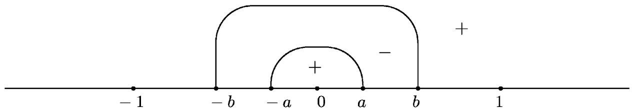

Fig. 7. Case 3a.

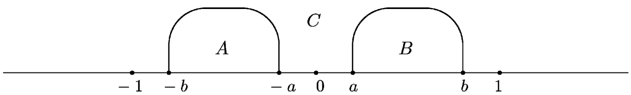

Fig. 8. Case 3b.

$$\begin{aligned}\lambda_3^{-1}\varphi_3(x) &= G_D\varphi_3(x) = E^x\int_0^{\tau_D}\varphi_3(X_t)\,dt\\ &= E^x\int_0^{\tau_\Omega}\varphi_3(X_t)\,dt + E^x\int_{\tau_\Omega}^{\tau_D}\varphi_3(X_t)\,dt\\ &= E^x\int_0^{\tau_\Omega}\varphi_3(X_t)\,dt + E^x\left(\left(\int_0^{\tau_D}\varphi_3(X_t)\,dt\right)\circ\theta_{\tau_\Omega}\right).\end{aligned}$$

By the strong Markov property this is

$$\begin{aligned}&E^x\int_0^{\tau_\Omega}\varphi_3(X_t)\,dt + E^x\left(E^{X(\tau_\Omega)}\int_0^{\tau_D}\varphi_3(X_t)\,dt; X(\tau_\Omega)\in D\backslash\Omega\right)\\ &= G_\Omega\varphi_3(x) + E^x(G_D\varphi_3(X(\tau_\Omega)); X(\tau_\Omega)\in D\backslash\Omega)\\ &= G_\Omega\varphi_3(x) + \lambda_3^{-1}E^x(\varphi_3(X(\tau_\Omega)); X(\tau_\Omega)\in D\backslash\Omega).\end{aligned}$$

It follows that for $x\in\Omega$ we have

$$\lambda_3 = \frac{\varphi_3(x)}{G_D\varphi_3(x)} = \frac{\varphi_3(x)}{G_\Omega\varphi_3(x) + \lambda_3^{-1}E^x(\varphi_3(X(\tau_\Omega)); X(\tau_\Omega)\in D\backslash\Omega)}. \tag{5.5}$$

We want to estimate $E^x(\varphi_3(X(\tau_\Omega)); X(\tau_\Omega)\in D\backslash\Omega)$. Since φ_3 is orthogonal to φ_1,

$$\int_D \varphi_3(y)\varphi_1(y)\,dy = 0.$$

Since φ_1 is positive, symmetric on $(-1,1)$, non-decreasing on $(-1,0)$ and non-increasing on $(0,1)$, we obtain that

$$\begin{aligned}0 &> \int_{-b}^{b}\varphi_3(y)\varphi_1(y)\,dy\\ &\geqslant \int_{-b}^{-a}\varphi_3(y)\varphi_1(a)\,dy + \int_{-a}^{a}\varphi_3(y)\varphi_1(a)\,dy + \int_a^b\varphi_3(y)\varphi_1(a)\,dy\end{aligned}$$

and it follows that

$$\int_{-b}^{b} \varphi_3(y)\,dy < 0.$$

Using the density for $P^x(X(\tau_\Omega)\in\cdot)$ given by (5.3) we see that for $x\in\Omega$ we have

$$E^x(\varphi_3(X(\tau_\Omega)); X(\tau_\Omega)\in D\backslash\Omega) = \int_{D\backslash\Omega} \varphi_3(y) f(y)\,dy, \tag{5.6}$$

where

$$f(y) = \frac{1}{\pi}\frac{(r^2 - |x - x_0|^2)^{1/2}}{(|y - x_0|^2 - r^2)^{1/2}|x - y|},$$

$r = (1-b)/2$ and $x_0 = (-1-b)/2$. Of course, f depends on x and y but x may be treated as fixed. We have $f(y) = cp(y)q(y)$, where $c = \pi^{-1}(r^2 - |x - x_0|^2)^{1/2}$, $p(y) = (|y - x_0|^2 - r^2)^{-1/2}$, $q(y) = |x - y|^{-1}$. We may assume that $x < -b < y < 1$. For such x, y we have

$$p'(y) = -(y - x_0)((y - x_0)^2 - r^2)^{-3/2} < 0,$$

$$p''(y) = (2(y - x_0)^2 + r^2)((y - x_0)^2 - r^2)^{-5/2} > 0$$

and

$$q'(y) = -(y - x)^{-2} < 0, \quad q''(y) = 2(y - x)^{-3} > 0.$$

Therefore

$$f''(y) = c(p''(y)q(y) + 2p'(y)q'(y) + p(y)q''(y)) > 0.$$

In other words, $f(y)$ is a convex function for $y\in(-b,1)$. Let $0\leqslant y<z<b$. We have $f(z) - f(y) = (z - y)f'(\xi)$, $\xi\in(y,z)$ and $f(-y) - f(-z) = (z - y)f'(\eta)$, $\eta\in(-z,-y)$. By convexity we see that $f'(\xi)\geqslant f'(\eta)$, so $f(z) - f(y)\geqslant f(-y) - f(-z)$. Therefore $f(z) + f(-z)\geqslant f(y) + f(-y)$ for $0\leqslant y<z<b$.

We know that φ_3 is symmetric, $\varphi_3(y)\geqslant 0$ for $y\in(-a,a)$ and $\varphi_3(y)\leqslant 0$ for $y\in(-b,-a)\cup(a,b)$. Hence

$$\int_{-b}^{b} \varphi_3(y) f(y)\,dy$$
$$= \int_0^a \varphi_3(y)(f(y) + f(-y))\,dy + \int_a^b \varphi_3(y)(f(y) + f(-y))\,dy$$

$$\leqslant \int_0^a \varphi_3(y)(f(a)+f(-a))\,dy + \int_a^b \varphi_3(y)(f(a)+f(-a))\,dy$$
$$= (f(a)+f(-a))/2 \int_{-b}^b \varphi_3(y)\,dy.$$

But we know that the last integral is negative and therefore

$$\int_{-b}^b \varphi_3(y) f(y)\,dy < 0.$$

Thus for $x \in \Omega = (-1, -b)$ we get by (5.6)

$$E^x(\varphi_3(X(\tau_\Omega)); X(\tau_\Omega) \in D \backslash \Omega) \leqslant E^x(\varphi_3(X(\tau_\Omega)); X(\tau_\Omega) \in (b, 1)).$$

Note also that for $x \in \Omega$ the denominator on the right-hand side of (5.5) is positive. Therefore for $x \in \Omega$ we get from (5.5)

$$\lambda_3 \geqslant \frac{\varphi_3(x)}{G_\Omega \varphi_3(x) + \lambda_3^{-1} E^x(\varphi_3(X(\tau_\Omega)); X(\tau_\Omega) \in (b, 1))}. \tag{5.7}$$

Let $||\varphi_3||_\Omega = \sup\{\varphi_3(x) : x \in \Omega\}$ and $x_* \in \Omega$ be such that $\varphi_3(x_*) = ||\varphi_3||_\Omega$. By symmetry, $\sup\{\varphi_3(x) : x \in (b, 1)\} = ||\varphi_3||_\Omega$. Putting $x = x_*$ in (5.7) we obtain

$$\lambda_3 \geqslant \frac{||\varphi_3||_\Omega}{||\varphi_3||_\Omega G_\Omega 1(x_*) + \lambda_3^{-1} ||\varphi_3||_\Omega P^{x_*}(X(\tau_\Omega) \in (b, 1))}$$
$$= \frac{1}{E^{x_*}(\tau_\Omega) + \lambda_3^{-1} P^{x_*}(X(\tau_\Omega) \in (b, 1))}. \tag{5.8}$$

By Lemma 5.2(i), formula (5.2) and the fact that $b \in [2/3, 1)$, we get $E^{x_*}(\tau_\Omega) \leqslant (1 - b)/2 \leqslant 1/6$ and

$$P^{x_*}(X(\tau_\Omega) \in (b, 1)) \leqslant \frac{(1-b)^2}{8\pi b^2} \leqslant \frac{1}{32\pi}.$$

Also, $\lambda_3 > \lambda_2 \geqslant 2$. Therefore the expression in (5.8) is no smaller than $(1/6 + 1/(64\pi))^{-1} > 5$. By (5.7) we get $\lambda_3 > 5 > 3\pi/2$, which gives a contradiction and completes the proof of the theorem. □

We conclude with a proposition providing some information for general λ_n's and general φ_n's in $D = (-1, 1)$.

Proposition 5.5. *Let $D = (-1, 1)$ and $n \in \mathbb{N}$. Then $\lambda_n \leqslant n\pi/2$ and φ_n has no more than $2n - 2$ zeros in D.*

Proof. The inequality $\lambda_n \leqslant n\pi/2$ follows from Theorem 3.14 and the fact that for $D = (-1, 1)$, $\mu_n = (n\pi/2)^2$, where μ_n are the eigenvalues for the Dirichlet Laplacian in D, problem (3.31).

For $x_0 \in D$ we will say that the Steklov function u_n changes the sign at $(x_0, 0) \in H$ if for each $r > 0$ the set $\{(x, t) \in H : (x - x_0)^2 + t^2 < r^2\}$ contains the points for which $u_n(x, t) < 0$ and the points for which $u_n(x, t) > 0$. By Proposition 3.17 and Theorem 4.1 if $\varphi_n(x_0) = 0$, $x_0 \in D$, then u_n must change sign at $(x_0, 0)$. If u_n changes sign at $(x, 0)$, $x \in D$ more than $2n - 2$ times, then u_n would have more than n nodal parts (we omit the details here). But this is impossible by Theorem 3.11. It follows that φ_n has no more than $2n - 2$ zeros in D as asserted by the proposition. $\square$

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